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MATTER MATTERS

Metaphysics and Methodology in the Early Modern Period

KURT SMITH

Kurt Smith is Associate Professor of Philosophy at Bloomsburg University of Pennsylvania.

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For Michelle Peguero, who has taught me that
there is more that matters than matter

Preface

This book is about the methods of analysis and synthesis, and how they underwrote what will be called the 'mathematization' of early modern physics. That being said, this book is not about mathematics. Rather, any appeal to mathematics in what follows is in service to my telling a related story—a story about metaphysics and methodology in the early modern period. Over the past several decades interpretations of Descartes's metaphysics have emerged as a result of using his views on natural philosophy, particularly his physics, as a guiding interpretative tool. This book uses instead his mathematics as the primary interpretative tool.

I began to think seriously about these matters during my participation in a 2000 National Endowment for the Humanities Summer Seminar, 'Descartes and His Contemporaries', which was conducted by Roger Ariew and Daniel Garber. I learned a great deal from them, and from many of my fellow participants. The idea that the project would take the form of a book arose one stormy afternoon, as Daniel Garber and I discussed things over lunch, on the rooftop of Boudreaux's, a restaurant that we NEH-types frequented while living in Blacksburg, Virginia. From this summer programme emerged an initial draft of what would become 'Descartes's Life and Works' in the *Stanford Encyclopedia of Philosophy*. Alan Nelson, the subject-editor with whom I worked (and a good friend), also began to encourage me to think of putting what I had to say in the form of a book. The idea was sinking in.

I participated in a 2004 National Endowment for the Humanities Summer Institute, 'The Intersection of Philosophy, Science and Theology in the Seventeenth Century', conducted by Steven Nadler and Donald Rutherford. I learned much from them, and from guest instructors and participants. This time, it was over coffee with Thomas Lennon, one of the institute's guest instructors, that the idea of a book project again came up. It was Thomas Lennon who finally convinced me that, instead of presenting my view piecemeal in a series of journal articles (which I had been doing up to this point), I would do better to present it in a single setting—a book. So, the book took on the status of a real project in the summer of 2004.

Philosophy books are always the product of the community of scholars. None arise in a vacuum, and even though the author is privileged with taking credit, it is clear that without a healthy, friendly community of dedicated scholars, there would be little to take credit for. This book is no exception.

Daniel Garber played a pivotal role in arranging a visiting fellow position for me at Princeton University during my spring 2007 sabbatical, graciously spending much time talking shop. It was during this sabbatical (and the summer to follow) that the bulk of the manuscript was written, the task lightened by my having spent the three previous years working out much of the book's structure and content.

About the ideas in this book, I have bent the ears of many scholars over the years. Several people stand out, and I would be remiss if I did not mention them by name. In addition to those already mentioned, I am indebted to Richard Arthur, Gregory Brown, Kenneth Brown, Jill Buroker, David Cunning, Dennis Des Chene, Corey Dyck, Geoffrey Gorham, Emily Grosholz, Helen Hattab, Benjamin Hill, Paul Hoffman, Douglas Jesseph, Nicholas Jolley, Henrik Lagerlund, Cathay Liu, Paul Lodge, Edwin McCann, Christia Mercer, Lawrence Nolan, Tad Schmaltz, Alison Simmons, Edward Slowik, Catherine Wilson, and Charles Young.

Some of the ideas in this book were presented at workshop-like talks. I am indebted particularly to audiences at Brown University (The New England Early Modern Reading Group), Calpoly San Luis Obispo, Claremont Graduate University, Rice University (the International Young Leibniz Conference), University of California at Irvine (The Cartesian Circle), Cornell University (the 4th Biennial Margaret Dauler Wilson Conference), University of North Carolina at Chapel Hill, and the University of Western Ontario (The Descartes Research Group).

I also thank the department of mathematics at Bloomsburg University of Pennsylvania for including me as a presenter in its colloquium series—and I thank several of the mathematicians in the department who continued to discuss with me this material, especially, William Calhoon, Drue Coles, Paul Loomis, John Polhill, and John Riley. I also thank my philosophy colleagues, Steven Hales, Gary Hardcastle, Wendy Lee, and Scott Lowe for their support and encouragement, and especially Richard Brook, who spent long hours over lunch, over several years, discussing all the gruelling details. And, I thank the late Schwimmers, Seymour and Ruth, for kindly bequeathing to me a large number of rare and important books from their library that were vital to my project.

Lastly, I am greatly indebted to Patricia Easton, Thomas Lennon, Steven Nadler, and Alan Nelson for looking carefully through early versions of the manuscript and for providing helpful remarks. And, I thank Michelle Peguero for composing the illustrations.

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Introduction

Why matter? According to seventeenth-century ontology, it is possible that there *be* something and yet this something not be material. A soul or a mind is an example of such a being. The Beatific Vision is but one depiction of such a cosmos: an eternal, immaterial, well-ordered hierarchical structure of souls that is directed at God: no doubt a medieval rendering of Plato's system of ideas—the eternal, immaterial system that is directed ultimately at the Good itself.[1] So, why matter? Why a spatio-temporal cosmos? This is not an empirical question, of course, since space and time together constitute the very conditions that make *possible* the empirical or sensible. The very thing for which we would be looking would be presupposed in the inquiry. No, it is a conceptual—a *philosophical*—question. The aim of this book is to look at a seventeenth-century answer to this question. The answer reveals an important connection that philosophers of the period were beginning to work out between a certain conception of matter and mathematics. It reveals the conceptual landscape that underwrote the success of the 'mathematization' of early modern physics, which I shall express as the twofold thesis:

> (T) The analysis of the material world inevitably leads to mathematics and mathematics inevitably takes matter as its object.

Our examination will focus primarily on the views of René Descartes (1596–1650) and Gottfried Leibniz (1646–1716), the former getting most of the attention.

Let me begin with something Descartes says in *Principles*, Part II, Article 8. There, he claims that, although we can conceive of the number *ten* or the quantity *ten feet* without attending to any specific body or specific region of what he calls *res extensa* (the extended thing), the number or quantity 'is unintelligible without some extended substance of which it is the [number or] quantity'.[2] Number, he says in Article 8, is only *conceptually* distinct from extended substance.[3] For Descartes what this meant was that, although one could think of number while *ignoring* extension or the extended substance, which was what one did when abstracting,[4] one could not conceive of quantity or number existing independently of extension or the extended substance. There is only a distinction in reason between them (between number and extension), but not in reality. This seems to commit Descartes to the conditional:

---

[1] By 'eternal' I mean non-temporal. For more on this concept, and the concept of *hierarchy*, see John Marenbon and D. E. Luscombe's 'Two Medieval Ideas: Eternity and Hierarchy', in *The Cambridge Companion to Medieval Philosophy*, edited by A. S. McGrade (Cambridge: Cambridge University Press, 2003), pp. 51–72.

[2] Henceforth, when citing Descartes I use *Oeuvres de Descartes*, Charles Adam and Paul Tannery (eds) (Paris: Vrin, 1964), and whenever possible I cite the English translation, *The Philosophical Writings of Descartes*, 3 vols translated by John Cottingham, Robert Stoothoff, and Dugald Murdoch (Cambridge: Cambridge University Press, 1984). I cite the Adam and Tannery (AT) volume and page numbers, along with the Cottingham, Stoothoff, and Murdoch (CSM) volume and page numbers. When citing vol. 3, I include Anthony Kenny (thus CSMK). The citation here, then, is: AT VIIIA, 45; CSM I, 226.

(M1) If mathematics is intelligible, matter exists.

By 'matter' here I mean what Descartes denoted by the phrase *res extensa*, a genuine substance whose nature is simply extension in length, breadth, and depth.[5] I intend (M1) to capture that part of (T) which tells us that mathematics inevitably takes matter as its object.

Descartes's commitment to (M1) is strengthened, I think, in a remark he makes to Mersenne in a 27 May 1638 letter.[6] Mersenne had apparently asked Descartes in a previous letter about the status of extension, on the assumption that extension was taken to be simply space. He wonders what ontological status Descartes will assign to this space, which Mersenne apparently took to be a kind of *nothing*, given that God does not create the universe. Of course, space is for Descartes nothing other than corporeality itself, and is no different in essence from an impenetrable solid body.[7] So, space will not be a nothing. Descartes says that 'not only would there not be any space, but even those truths which are called eternal—as that "the whole is greater than its part"—would not be truths'.[8] In light of the interpretation to be worked out in what follows, I read Descartes's answer to Mersenne here as telling us essentially that had God not created matter (that is, extension) there would be no mathematical truths.

Although much more will have to be established before making this perfectly acceptable, we might initially reconstruct the line of reasoning as follows: if mathematics is intelligible, matter exists. Assuming that matter does not exist, it follows (by *modus tollens*) that mathematics is unintelligible. Now, given that the intelligibility of mathematics is coextensive with its possibility, in a world in which mathematics is unintelligible, mathematics is also impossible. Descartes holds that 'whatever is true is something'.[9] Thus if, in being impossible, mathematics were a kind of nothing, there would be no mathematical truths. And so, in a world in which there is no matter, there are no mathematical truths. I will show that for Descartes, Leibniz, and others in the period, then, now casting (M1) in terms perhaps more familiar to the reader, the existence of matter is taken to be a necessary condition for the intelligibility of mathematics.

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[3] AT VIIIA, 45; CSM I, 226.

[4] See, for example, AT X, 440; CSM I, 58 and AT IV, 120; CSMK III, 236.

[5] Following Descartes, this is what I shall mean by 'matter' throughout the remainder of this book. In places where it seems needed, I will flag the reader by inserting parenthetical reminders.

[6] AT II, 138; CSMK III, 103.

[7] See, for example, *Principles*, Part II, Article 11 (AT VIIIA, 46; CSM I, 227).

[8] AT II, 138; CSMK III, 103.

[9] AT VII, 65; CSM II, 45.

On the flipside of this, I argue that Descartes (and Leibniz, and others) held that there is something about the existence of *res extensa* that is sufficient for guaranteeing the intelligibility (or possibility) of mathematics. This would commit Descartes (and the others) to holding:

(M2) If matter exists, mathematics is intelligible.

I intend (M2) to capture that other part of (T) which tells us that the analysis of the material world inevitably leads to mathematics. The bulk of the texts supporting (M2) are found in the context of seventeenth-century combinatorial theory, which includes, but is not limited to, the work of Leibniz. And although Leibniz did not take what I am calling 'matter'—that is, extension in length, breadth, and depth—to be substantial, he did take *body* to be. As we will see in Part III, a look at Leibniz's later work will show that matter, as Descartes understood it, was taken to be metaphysically *prior* to body. For Leibniz, again in respect to his later work, matter (or Cartesian extension) emerges from the monads as a nexus of relations. According to one reading of Leibniz to be considered, body (and we could add here, number and quantity) emerges as the result of an informed division of extension, where being *informed* introduces the *combinability* of one body with another. If, as mathematicians agree, the combinatorial nature of body is sufficient for guaranteeing the intelligibility of mathematics (that is, the combinatorial nature of body instantiates the conditions that underwrite mathematics), and body *presupposes* matter, then if there is body (and thus matter), the intelligibility (or possibility) of mathematics is guaranteed. But, more on that later.

Now, if we conjoin (M1) and (M2), we get the biconditional:

(M3) Mathematics is intelligible if, and only if, matter exists.

To be sure, (M3) is not the only important claim that can be drawn from (T). And, I suspect that some scholars may find my use of the contemporary logical machinery (the biconditional, for instance) artificial, for seventeenth-century philosophers, especially Descartes, spoke more about the relations of ideas than about propositions (though even Descartes employs proposition-talk on occasion). Be that as it may, (M3) emphasizes the logical or conceptual connection that I want to explore, a connection between the intelligibility of mathematics and the existence of matter. The connection would allow Descartes, Leibniz, and others to hold the view that God's creating a mathematical reality is the same thing as his creating a material one. On this view, matter is not taken to be a divine afterthought or extravagance, but instead is taken to serve as *the* metaphysical medium by way of which God instantiates mathematical truth and meaning. Matter plays an essential metaphysical role in the cosmos. And so, as the title of this book asserts: matter matters.

Plotinus (204–270 CE) developed two theoretical conceptions of matter that are relevant to our study of seventeenth-century rationalist theories of matter, which Christian Schäfer calls *inferior*-matter and *material*-matter. Material-matter is the matter we sense.[10] By contrast, inferior-matter is 'the amorphous *hypokeimenon*', and is for Plotinus matter proper.[11] It is 'what we *ex post* identify as the ("in itself") structureless "fabric", which underlies material-matter as we know it of the bodily universe'.[12] In being the *hypokeimenon*—here the underlying extended subject (or corporeal substance)—the term 'inferior' is misleading. Perhaps a better term would have been *underlying*-matter. It is, I think, akin to the receptacle of Plato, as introduced in his *Timaeus*. But, more importantly for the view to be worked out here, it is the matter of Descartes, which Emily Grosholz identifies as the 'unbounded monolith', standing metaphysically prior to the individuated bodies we observe in the cosmos.[13] In the examination to follow, and with clarity as a principal aim, I shall use the term 'matter' to refer to *res extensa* and *extensio*, the nature of which is extension in length, breadth, and depth, and the terms 'individuated body' to refer to that which arises from matter upon matter's being informed (divided). The term 'body' will be used to refer to both (*res extensa* and individuated body). From this point forward, *body* and *individuated body* will be taken to metaphysically presuppose *matter*, but not the other way around. When appropriate I shall remind the reader of this technical distinction by way of parenthetical remark.

This distinction becomes useful when the discussion turns to Leibniz. For, although he took body to be substantial, he did not take extension to be. So, in terms of the distinction, he did not take what I call *matter* to be substantial. Even so, both Leibniz and Descartes held that the *bodies* we experience are substantial (or real) in a sense that is important to their respective metaphysical systems. So, we can work out a way to accommodate both philosophers, while at the same time being mindful of their differences. Our examination will be more productive in the end, I believe, if we make available a technical terminology, as I have done, that allows us to navigate a clear course between Descartes and Leibniz (and between several others living in the period).

The above Neoplatonist distinction (the distinction drawn between *matter* and *body*) will work because rationalists (Descartes and Leibniz included) of the period adopted it. But a warning is in order. Although I believe that seventeenth-century rationalists adopted certain features of Neoplatonism, which is why Plotinus's view will have some weight in the examination to follow, they certainly did not

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[10] Christian Schäfer, 'Matter in Plotinus's Normative Ontology', *Phronesis* XLIX, 3 (2004), pp. 266–94.

[11] Ibid., p. 270.

[12] Ibid.

[13] Emily Grosholz, *Cartesian Method and the Problem of Reduction* (Oxford: Clarendon Press, 1991), p. 66.

Is matter substantial? Is it already individuated when it is conceived as a substance?

Ch. 1. 'matter

adopt every feature. This has been a source of contention between scholars. Ernan McMullin, for instance, commits Descartes to much more of the Neoplatonist programme than is warranted in my opinion, reading Descartes as holding, for example, that mathematical truth does (and, in fact, *must*) exist independently of matter.[14] But, if Descartes held (M3), as I contend, then mathematical truth (which presupposes the intelligibility of mathematics) and the existence of matter are not independent of one another. In every world in which mathematics is intelligible, matter exists, and vice versa.

Even so, I think that McMullin is right when he claims, 'After Aristotle himself, there is probably no more crucial figure in the entire story of the concept of matter than Descartes.'[15] And, I think that he is right in his assessment of Leibniz's importance to the story for, concerning problems surrounding matter, substance, and individuation, 'the only metaphysician of the century for whom it [the individuating of material substances] remained a major issue, almost *the* major issue, was Leibniz'.[16] Seeing Descartes and Leibniz as important figures in the story of modern matter is not new, of course, for as McMullin points out, it goes back to at least Newton, who was continually engaged in trying to distinguish his conception of matter from those of Descartes and Leibniz.[17]

Metaphysical concerns about the existence of matter can be found at every turn in the period. In Dialogue IX of Malebranche's *Dialogues on Metaphysics*, for instance, we find Aristes asking: 'But, Theotimus, since God is sufficient unto Himself, why the plan of creating the world?'[18] Here, *the world* includes the spatiotemporal or *material* cosmos, the nature of which is in stark contrast to the immaterial, eternal nature of God. Like other philosophers of the period, Aristes is cast by Malebranche as holding a form of Neoplatonism.[19] At this place in the dialogue, when he asks the question about the world, Aristes appears to be concerned mostly about how one is to make sense of God's desiring or willing, since to desire or to will implies that God is lacking in some respect. But, entangled in this concern, it seems, is the worry over matter—for, given that we could make sense of God's willing, we would still be faced with the question of why he desires or wills the existence of *matter*. As Aristes might put it, why does God will the existence of that which is repugnant to his own nature? Theodore and Theotimus, who are Aristes's theological interlocutors, ultimately answer this question within the framework of a theodicy. For, they seem to take the question '*Why does God create matter?*' to be connected to the question about

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[14] *The Concept of Matter in Modern Philosophy*, edited by Ernan McMullin (Notre Dame: University of Notre Dame Press, 1978), p. 17.

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[15] Ibid., p. 14.

[16] Ibid., p. 22.

[17] Ernan McMullin, *Newton on Matter and Activity* (Notre Dame: University of Notre Dame Press, 1978), pp. 29–30.

[18] Nicolas Malebranche, *Dialogues on Metaphysics*, Dialogue IX, translated by Willis Doney (Abaris Books, 1980), p. 201.

[19] Aristes appears to be representative of those who hold the doctrine of emanation. 'We are', he says, 'a necessary emanation of the Deity.' Dialogue IX, (Doney), p. 199.

This q. must be answered Xlogically, not
as a theodicy — The latter is the

the existence of evil. And, given that Theodore represents Malebranche's view, it would be safe to say that this is how Malebranche thought it best to deal with the question.

However, unlike Malebranche, Descartes does not appear to make the connection between the problem of matter and the problem of evil.[20] Along Augustinian lines, Descartes sees evil as a kind of privation.[21] But, matter is a different story altogether. As pointed out earlier, Descartes sees matter as an existent something, which, when taken generally, he says, is an incorruptible *substance*.[22] He goes on to say, 'if nature [that is, the natural world] is considered in its general aspect, then I understand by the term nothing other than God himself, or the ordered system of created things established by God'.[23] So, God and the material cosmos are intimately related for Descartes. This view is not unique to Descartes, of course. Newton, for instance, would come to hold something similar, where God himself 'constitutes duration and space'[24] and it is 'in Him [that] all things [are] contained and moved'.[25] So, if matter is something created and continually maintained by God, and matter is connected to evil, God must be understood as having created and as continually maintaining evil. But, Descartes holds the consequent of this conditional to be false,[26] from which he would surely derive the falsity of its antecedent.

X

Even so, the question is still unanswered: why does God create matter? For, as was mentioned above, certainly an omnipotent God could have created an immaterial cosmos. In fact, according to Christian dogmatic and apologetic texts, such a cosmos is exactly the sort that God ultimately desires. This suggests another way of raising the question about matter, a way that was available to Descartes and to others of the period.[27] As already noted, the Beatific Vision, famously discussed by Aquinas and Dante, suggests that God's endgame includes the annihilation of all matter, along with the annihilation of Satan and his followers, where God exists as the focal point of a purely spiritual (that is, incorporeal) hierarchical structure of souls. This communion (of souls with God) is supposed to be *eternal* and *incorporeal*, which does more than to suggest that it is both non-temporal and non-spatial.[28] Construing the endgame as that which God desires (his *aim*), we run into a problem if we consider a second doctrine—the doctrine of divine simplicity. This doctrine tells us that God is simple (as opposed to composite or complex), and acts in the most efficient way.

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[20] For a brief note about the divorce between matter and evil, see McMullin (*The Concept of Matter* 1978), p. 20.

[21] AT VII, 54; CSM II, 38.

[22] See, for example, AT VII, 14; CSM II, 10.

[23] AT VII, 80; CSM II, 56.

[24] Isaac Newton, *The Principia*, translated by Andrew Motte (Amherst: Prometheus Books, 1995), Book III, p. 441. McMullin finds a similar view in Newton's *De Gravitatione*, (*Newton on Matter and Activity* 1978), p. 76.

[25] Ibid., p. 441.

[26] AT VII, 53–4; CSM II, 37–8.

[27] Whether anyone actually raised the question this way is irrelevant to my point: it is a way of raising the question that was clearly available to Descartes and to others. No doubt, Descartes seems able to sidestep the question, if it had been raised this way, since he rejects our being able to know the final end (i.e., the endgame) of God's plan for the world. But, since this particular depiction of the endgame is taken to have been the product of revelation, Descartes might in the end have accepted it as something to be considered. See, for example, the August 1641 letter to Hyperaspistes (AT III, 431; CSMK III, 195), and the *Conversation With Burman*, 16 April 1648 (AT V, 158; CSMK III, 341).

The Gks cd. never escape the puzzle
of why matter is, as well as what it
we
[illegible] moral + one [illegible]

If the cosmic endgame is as Aquinas and Dante describe it, then God's creating a material cosmos would appear only to prolong God's getting what he desires. For, the efficient act would presumably be God's creating solely (and immediately) that incorporeal hierarchical structure. As with most things theological, there is a further kink in the story. After the fall of man, God apparently gives Satan dominion over the material world.[29] This certainly appears to be an obstacle to God's getting (that is, immediately getting) his desired end—that incorporeal hierarchical structure. Thus, in creating matter, and going so far as to surrender it to Satan, God appears not to have acted in the most efficient way. To resolve this apparent conflict between the Beatific Vision, the doctrine of divine simplicity, and the cosmogony of the material world, we need to answer the question: *why matter?* This may seem like a theological question, since it arises from a consideration of certain theological doctrines, but it is the sort of question that philosophers of the seventeenth century would have taken to fall squarely within the domain of philosophy.[30]

In connecting the intelligibility of mathematics to the existence of matter, Descartes and others in the period put themselves in a position to hold that the *necessity* attached to mathematical truth extends (we might think of its being extended by way of the biconditional) to the existence of matter. Thus, as I stated earlier, in every world in which mathematical truth holds, matter exists, and vice versa.

To set things out as clearly as possible I should say something about how I plan to approach this subject. This book is a philosophical study not only of the concept of matter, but of the constellation of concepts developed in the seventeenth century that form around this central concept. I should emphasize that it is not solely an historical study of the development of those philosophical ideas. Taking the history of philosophy to constitute a continuum of scholarly work, where history is found at one end and philosophy at the other, it is easy to see that some work is more historical than philosophical, and vice versa. Will Durant's work, I think, is an example of the strongly historical sort.[31] The works of Henk Bos, Emily Grosholz, and Matthew Jones are examples of more recent studies of Descartes that fall on the continuum close to Durant's.[32] By contrast, Gilbert Ryle's work is an example of the strongly philosophical sort.[33] Joseph Almog's two studies of Descartes are examples of more recent scholarship that fall close to Ryle's.[34] Both camps (the one encompassing Durant, the other Ryle) have said things about Descartes, but Durant's camp reveals little of interest philosophically, and Ryle's reveals little of interest historically. But, more importantly, because they lie towards the extreme ends of the continuum, neither gives us the best picture of *Descartes's* philosophical work. On my view, the best history of philosophy is reflected in work that falls close to the centre of the continuum. In light of this, I have tried to write this book with my sights aimed towards the centre.

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[28] I again refer the reader to Marenbon and Luscombe (2003).

[29] This is a highly contentious claim, of course, but it was something commonly believed by Christians of the period. Passages that suggest Satan's dominion over the material world were gleaned, for example, from 2 Corinthians 4:4 and John 12:31.

[30] See, for example, Evans, G. R. *Philosophy and Theology in the Middle Ages* (New York: Taylor & Francies, 2003), pp. 10–12.

[31] See, for instance, Will Durant's *The Story of Philosophy* (Garden City: Garden City Publication Co., 1943).

Matter resists our knowing
as well as our willing:
It impedes clear thought
+ virtuous action
obscures + distorts.

R. Harré drew a further distinction between two sorts of philosophical projects that he noticed were commonly pursued in the doing of history of philosophy.[35] The first he calls *critical philosophy* and the second *constructive philosophy*. Critical philosophy, he says, deals primarily with the working out of conceptual puzzles, the currency of exchange here being mostly in the form of arguments.[36] This, as I understand it, is nothing less than the doing of analytic philosophy as it is traditionally done. Constructive philosophy emerges as a kind of experiment, when after having worked up what one thinks an historical figure means by this or that, one tries to show how this fits into other things he said, perhaps resolving difficulties that previous scholars have raised.[37] Ultimately, critical and constructive philosophical projects are *conceptual* investigations. In addition to my meeting the rigorous scholarly (historical) demands of a project like this, I shall employ both the critical and constructive philosophical approaches in my working out the sense in which Descartes and Leibniz understood (M3) to be true. Fortunately, there is much in the way of scholarly work that will help me accomplish this.

In 1962 Toulmin and Goodfield expressed trepidation in putting together *The Architecture of Matter* on the grounds that the story of the evolution of the concept of matter was not only vast but largely unfamiliar to scholars.[38] 'To the best of our knowledge,' they write in the book's forward, 'this is the first recent attempt to give a coherent general account of the whole field we have called "matter-theory" . . . as it has evolved since the beginning of science.'[39] The book begins with the ancient Greeks, principally with Plato and Aristotle, and in relatively short order provides the promised general account. A year before this, in 1961, a conference at Notre Dame spawned several excellent papers that worked out some of the details of the development of the concepts of matter, but those papers that focused on the ancient and medieval periods would not be published until 1963,[40] and those that focused on the modern period (several of them to be commissioned later) would not see print until 1978.[41] Subsequently, it was Harré's *Matter & Method*,[42] published in 1964, which first provided some of the philosophical details of seventeenth-century theories of matter. Harré would follow-up on this material, later editing a collection of essays related to science and the concepts of matter, which appeared in 1986.[43] Though this is not an exhaustive list, the most recent books of both historical and philosophical significance that stand out are Daniel Garber's *Descartes' Metaphysical Physics*,[44] which was published in 1992, and the two-volume set (specifically Volume I, Section IV) that he and Michael Ayers edited, *The Cambridge History of Seventeenth-Century Philosophy*,[45] published in 1998.[46] The most recent monograph dedicated to this subject (at the time of writing this book) comes from Thomas Holden, which resurrects the Toulmin and Goodfield title, *The Architecture of Matter*, which was published in 2007.[47]

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[32] Matthew L. Jones, *The Good Life in the Scientific Revolution* (Chicago: Chicago University Press, 2000); Henk J. M. Bos, *Redefining Geometrical Exactness* (New York: Springer, 2000); Emily R. Grosholz, *Cartesian Method and the Problem of Reduction* (Oxford: Clarendon Press, 1991).

[33] Gilbert Ryle, *The Concept of Mind* (London: Hutchinson's University Library, 1949).

[34] Joseph Almog, *What Am I? Descartes and the Mind-Body Problem* (Oxford: Oxford University Press, 2002); *Cogito? Descartes and Thinking the World* (Oxford: Oxford University Press, 2008).

[35] R. Harré, *Matter & Method* (New York: St. Martin's Press, 1964), p. ix.

[36] Ibid.

[37] Ibid.

[38] Stephen Toulmin and June Goodfield. *The Architecture of Matter* (New York: Harper & Row, 1962), p. 15.

The unique contribution of *Matter Matters* will be its exploring the connection between matter and mathematics by way of a close examination of two interrelated methods of philosophizing found in the seventeenth century, namely, *analysis* and *synthesis*. The direct connection among these things—matter, mathematics, and the methods of analysis and synthesis—has thus far gone unexplored. As I hope to show, the methods constituted an important part of the conceptual groundwork that established what I call the 'mathematization' of modern physics. In plain terms, the methods of analysis and synthesis are what account for the success of the mathematical physics that emerged from the early modern period.

In Part I, I work out some preliminaries. This will help to set the stage for better understanding what I think is the intellectual context from which the modern conception of matter emerged. Although textually rooted, I take some liberty in establishing what I think was a view of the ancients (and medievals) that was in the air during the seventeenth century. This will require us to consider texts not included in university curricula, though available and widely discussed both in published books and in private correspondence. Specifically, I focus on Plato's divided line analogy introduced in the *Republic*, on his method of analysis found in various dialogues, and on the story of creation in his *Timeaus*. These things are set alongside Aristotle's method of classification. I show in what ways Plato's method of analysis looks to be connected to Aristotle's theory of classification. The works of Augustine, Eriugena, Lull, and Aquinas make brief appearances. And though only briefly considered, I take them (and other works briefly considered) to have been important works that served to intellectually bridge the two-thousand year divide between seventeenth century and ancient Greek philosophy. Part I prepares the way for the other Parts to follow, and so I encourage the reader to not skip over it.

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[39] Toulmin and Goodfield (1962), p. 15.

[40] McMullin *The Concept of Matter* (1978) tells of the 1961 conference in his preface.

[41] Ibid., preface. [42] Harré (1964).

[43] *The Physical Sciences Since Antiquity*, R. Harré (ed.) (New York: St. Martin's Press, 1986).

[44] Daniel Garber, *Descartes' Metaphysical Physics* (Chicago: The University of Chicago Press, 1992).

[45] *The Cambridge History of Seventeenth-Century Philosophy*, 2 vols, Daniel Garber and Michael Ayers (eds) (Cambridge: Cambridge University Press, 1998).

[46] There is also Grosholz's most recent book, *Representation and Productive Ambiguity in Mathematics and the Sciences* (Oxford: Oxford University Press, 2007).

[47] Thomas Holden, *The Architecture of Matter* (Oxford: Oxford University Press, 2007).

Part II primarily focuses on Descartes's method of analysis. This method, I contend, is his method for arriving at clear and distinct ideas. It is a method for establishing what I call the *master enumeration*, which is something like a schematic blueprint of the system of clear and distinct ideas. Here, the method of enumerating (or classifying) is shown to have played an important role in the development of the modern concepts of *measure* and *order*, which are related to the concept of *quantity*, and are, as Descartes writes, 'the exclusive concern of mathematics'.[48] Some of the view worked out here can be found in an earlier article of mine, 'A General Theory of Cartesian Clarity and Distinctness Based On the Theory of Enumeration in the *Rules*', *Dialogue* (2001). Part II goes far beyond what I argued in this article, though some things remain. For instance, the master enumeration is understood to be a logical partition, produced by an application of equivalence relations—or as Descartes will call them, common natures. I would like the reader to take from Part II an understanding of how Descartes's method of analysis is similar to Plato's, the latter having been developed in Part I. What is more, the reader must take with him or her from Part II an understanding of how the partitioning works, how the master enumeration is established, and how partitioning is related to the concepts of *clarity* and *distinctness*, in order to better appreciate the development of the method of synthesis to follow.

Part III focuses on the method of synthesis. Some of the ideas in Part III were first published in a short article of mine, 'Was Descartes's Physics Mathematical?' in *History of Philosophy Quarterly* (2003). Both Descartes and Leibniz are discussed in Part III, though Leibniz's view takes the position of front and centre. As Leibniz developed it, the method of synthesis emerges as a method of combinatorics, which played an important role in the development of equation making. We see this even in Descartes. To use a term that would arise in mathematics a century after Leibniz, the conceptual (or logical) systems of combinatorics that result from synthesis are *groups*. Here, I work up mathematical structures that lead up to a group, namely, those of the *semigroup* and the *monoid*. The group, as we shall see, constitutes the very conditions underlying all of mathematics. For example, the group structure is what underwrites addition and multiplication. We shall explore Leibniz's, and even Pascal's, fascination with combinatorics and their initial insights into the connection between permutations and the foundations of arithmetic. Here, other great mathematicians, such as Euler and Cayley, enter into the discussion. I also look at structures constituted of combinations of groups and monoids: the *Boolean algebra*, the *ring*, and the *field*. The latter two structures are what account for the success of Leibniz's and Newton's handling of motion.

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[48] AT X, 378; CSM I, 19.

In Part III, Chapter 12 (and in my closing remarks), we learn that the conceptual system that is established by analysis is isomorphic to a synthetic system—that is, a system meeting the group criteria. The equivalence relation (or common nature, as Descartes calls it) that serves to 'divide' (or sort) the objects of inquiry into the atomic categories (of the system) when performing analysis is nothing short of the binary operator we find at work in the synthetic system. Given that this latter system is a group, which is a mathematical system, and the system established by analysis is isomorphic to it, then the system established by analysis is a mathematical system. Thus, analysis and synthesis, understood as meeting the criteria of a group, are what conceptually account for the 'mathematization' of early modern physics. So, it is by way of these methods that seventeenth-century thinkers succeeded in producing a conceptual landscape that made possible a genuinely mathematical physics.

Part IV deals with a set of philosophical problems that arise at different places in Parts II and III. Primarily, the discussion attempts to resolve these problems, focusing on several arguments made by Descartes related to establishing the conditional: *If mathematics is intelligible, matter exists*. The second conditional, *If matter exists, mathematics is intelligible*, is, I believe, sufficiently established by the close of Part III. The reader will also find a critical examination of several relatively recent views attributed to Descartes, specifically the views of direct realism, modal actualism, and what might be called proto-transcendentalism. I focus on these views because they include insightful treatments of the arguments for the 'reality' of matter.

I suggest a way of understanding Plato's divided line analogy, which correlates with certain distinctions made by Kant, namely, the *Empirical Ideality-Reality* distinction, and the *Transcendental Ideality-Reality* distinction. They align almost perfectly with Plato's two great divisions of the line—the *Sensible* section and the *Intelligible* section, respectively. Here, we look at several interpretations that focus on Descartes's views concerning matter, which initially appear to be at odds with one another. One camp of interpreters, we will see, advocates for Descartes's holding the view that many bodies (corporeal substances) exist independent of the mind. Another camp advocates for Descartes's holding the view that only one body (*res extensa*) exists independent of mind. Individual bodies (such as the sun) are ideal according to this reading. In light of the Platonic-Kantian categories developed, we learn that both camps can be understood as successfully tracking important aspects of Descartes's view on matter. The one camp (that advocates for the 'many bodies' view) appears to be tracking an empirical realist view in Descartes, the other camp (that advocates for the 'one body' view) appears to be tracking a transcendental idealist (or even realist) view in Descartes.

It might come as some reassurance to the reader that, although certain mathematical concepts are dealt with in some detail, including those structures mentioned above, I have geared the discussions in Parts II and III so that a proficiency level of high-school maths will be sufficient for following the narrative—at any rate, readers should find the technical discussions easier to follow than, say, those offered by Grosholz and Bos in their authoritative works dealing with Descartes and mathematics.[49] My aim has been to include as part of the philosophical story the mathematics under discussion. The mathematics should not be seen as an obstacle to following the narrative. Thus, I encourage the reader to look at a particular table or equation under discussion as part of the story.

Dividing the book into four parts was a way to organize the main argument of the book—the aim of which is to establish (M3). Now, even though the book is divided into four parts, the chapters are continuous. This is because the narrative of the story runs the length of the entire book. The reader will run the risk of losing important parts of the story (and important parts of the main argument) if he or she skips around, or reads the chapters piecemeal. There is a continuous line of reasoning, from beginning to end, and I encourage the reader to let the narrative unfold as it is presented in the book. If the reader does this, by the book's end, he or she should better understand why seventeenth-century thinkers, such as Descartes and Leibniz, believed that matter matters, and this will include a better understanding of the conceptual landscape in place during the period that made possible the 'mathematization' of modern physics.

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[49] Grosholz (1991); Bos (2000).

PART I

PRELIMINARIES: THE CONTEXT OF MODERN MATTER

1

The Visible and the Intelligible

In Book VII of the *Republic* Socrates tells Glaucon that sensation, when presenting what appears to be a contradictory state of affairs, provokes the soul into summoning 'calculating reason',[1] for the purpose of resolving the apparent contradiction. 'The [sensory] experiences that do not provoke thought,' he says, 'are those that do not at the same time issue in a contradictory perception.'[2] Socrates calls those sensations that have this effect on the soul—that is, the effect of provoking thought—'provocatives' (*parakalounta*).[3] He considers a case of someone's holding up three fingers. As fingers, there is nothing presented that appears contradictory—'each appears to be equally a finger' (*daktulos men auton phainetai homoios hekastos*).[4] But, the ring finger appears to be shorter than the middle finger, and at the same time appears to be longer than the pinky finger. Thus, this sensory experience presents the ring finger as being both short and long—or better put, it initially presents the finger as possessing simultaneously what appear to be contrary properties. This is a provocative.

Calculating reason goes to work to determine whether what is being sensed is *one* or *two*.[5] Although it is not perfectly clear what this means, Socrates seems to have in mind that in understanding each as being equally fingers, we understand the sensory experience to be presenting *one* (kind of) thing, namely, *finger*.[6] But long and short (or great and small) are not one (kind of) thing, they are understood to be two distinct (kinds of) things (which, in this case, are opposites). Sensation appears to confound them by presenting the ring finger as an instance of both. Calculating reason determines the sense in which there are really *two* (kinds of) things in play. 'And for clarification of this,' Socrates says, 'the intelligence is compelled to contemplate the great and small, not thus confounded but as distinct entities, in the opposite way from sensation.'[7]

What calculating reason determines is that it is in respect to the middle finger that the ring finger is short (but not long), and in respect to the pinky finger that the ring finger is long (but not short). Thus, long and short, though both are possessed by the ring finger at the same time, are not possessed *in the same respect*. Understanding things this way dismantles the apparent contradiction originally presented by sensation. Even so, the provocatives of sensory experience are important. This is so because the sort of sensory experience that provokes such intellectual activity, Socrates says, is the origin of our even asking such questions as, 'What is the great and the small?'[8] Such experience also plays an important role in Socrates's (or, perhaps more accurately, Plato's) rationalist epistemology and ontology. For, this way of conceiving the provocation of thought (or activation of intelligence), Socrates notes, 'is [also] the origin of the designation *intelligible* for the one, and *visible* for the other.'[9] In other words, this is what famously leads him to talk of two categories of object: the *intelligibles* and the *visibles*. It should not go unnoticed that this view also suggests that were it not for the provocatives of sensory experience, human beings might not *think* at all.

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[1] *Republic*, 523b–524e. In a footnote, Shorey explains his translation of '*logismon*' as 'calculating reason' by an appeal to Plato's *Parmenides*, where Plato uses the phrase '*tois logismo lambanomenois*', p. 157, fn. d. Plato. *Republic*, translated by Paul Shorey, in *Plato: The Collected Dialogues*, edited by Edith Hamilton and Huntington Cairns (Princeton: Princeton University Press, 1989), pp. 575–844.

[2] Ibid., 523b.

[3] Ibid., 523c.

[4] Ibid., 523c–d.

[5] Ibid., 524b.

[6] Or, fingerhood.

[7] *Republic*, 524c.

The remark about the origin of the distinction between intelligibles and visibles seems a bit strange coming in Book VII, for it is not obviously connected to the allegory of the cave, which was introduced at the Book's opening. The remark about the origin of the distinction, I think, is related to an analogy that Socrates had introduced at the close of Book VI: the analogy of the divided line (I take the two analogies to be connected, but the connection is not obvious). The analogy of the divided line influenced philosophers of the seventeenth and eighteenth centuries. For instance, it involved thinkers of the period in considering the philosophical underpinnings of a distinction between the faculties of sense and intellect (whose objects are the visibles and the intelligibles, respectively), a distinction that would work its way up to the modern period by way of philosophers of the earlier medieval and scholastic periods.[10] What is more, it is an important source, whether directly or indirectly, of the philosophical concepts of *image*, *object*, *representation*, *idea*, *abstraction*, *universal*, *mathematical object*, *opinion*, and *knowledge*, to name just a few. As I suggested, the analogy treats simultaneously of a rationalistic metaphysics and epistemology.[11] In preparation for our examination of the modern period, there is much value in looking, even if only briefly, at the analogy of the divided line. So, I will examine it as it is introduced at the close of Book VI of the *Republic*.

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[8] *Republic*, 524c. [9] Ibid., 524c.

[10] Two excellent volumes track the historico-philosophical development of faculty psychology, the differences between sensation and intellection, the concepts of intellection versus will, and so on, through the medieval and scholastic periods, including the early modern period—one is *Logic and the Workings of the Mind*, edited by Patricia A. Easton, vol. 5 of the series North American Kant Society Studies in Philosophy (Atascadero: Ridgeview Publishing, 1997); the other is *Emotions and Choice From Boethius to Descartes*, edited by Henrik Lagerlund and Mikko Yrjonsuuri (Dordrecht: Kluwer, 2002).

[11] Though with respect to the idea that human beings would not think were it not for sensation, the view arguably can be cast as a form of empiricism as opposed to a form of rationalism.

ANALOGY OF THE DIVIDED LINE

After drawing a distinction between visible and intelligible objects, Socrates says, 'Represent them then, as it were, by a line divided into two unequal sections . . . '[12] The inequality of the sections is significant, for with respect to epistemic states their magnitudes will represent their 'degrees' of clarity,[13] the intelligible being 'more clear' than the visible; and with respect to the objects of knowledge their magnitudes will represent their 'degrees' of truth or reality, intelligible objects being 'more true' or 'more real' in some sense than their visible counterparts.[14] So, the line initially looks thus:

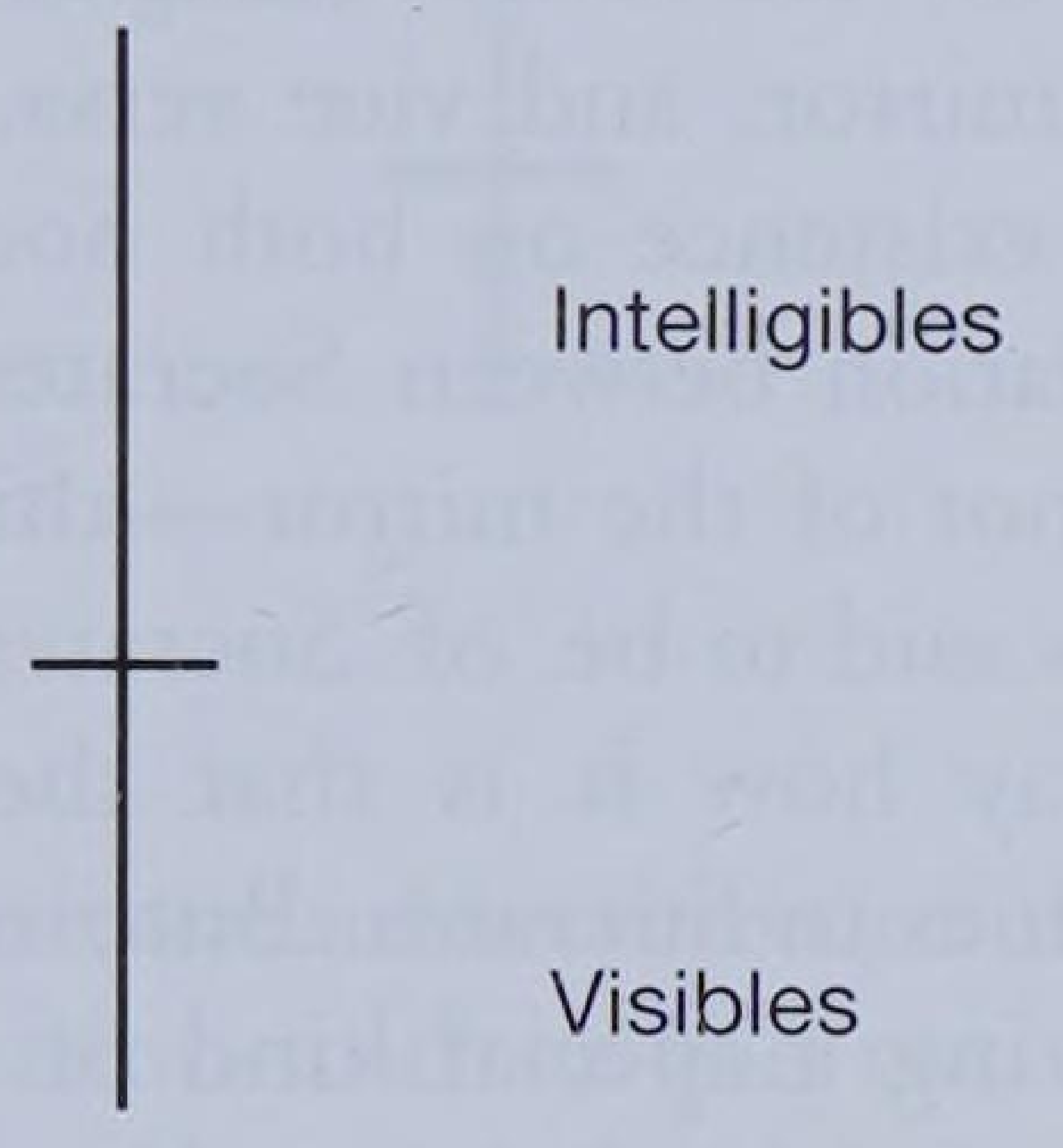

Figure 1.1

He says, 'cut each section again in the same ratio', which produces two line sections in each initial section, making four sections in total:

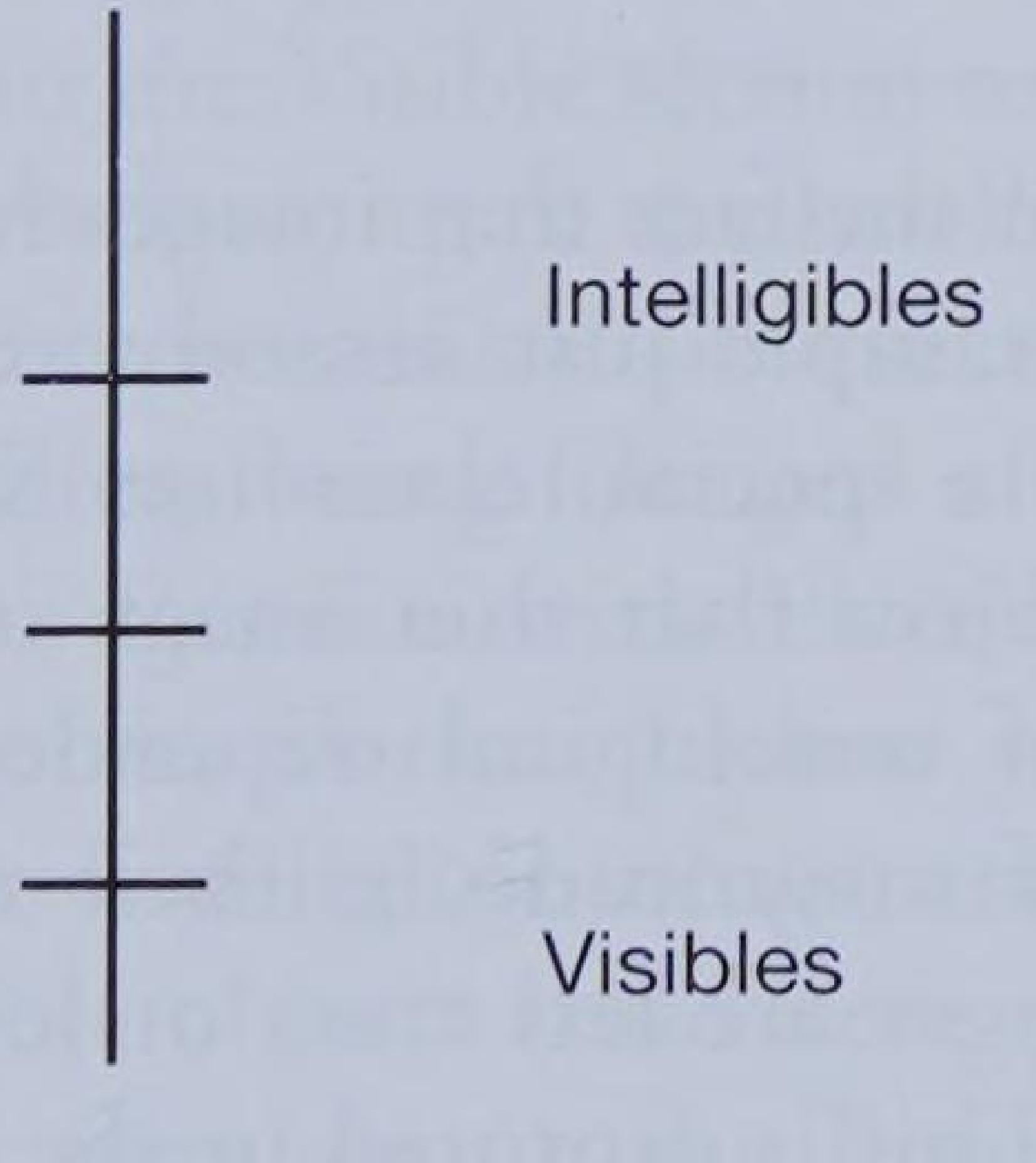

Figure 1.2

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[12] This passage is from (Shorey) p. 745 (509d). In my presentation of the analogy, I follow the analysis of Nicholas D. Smith, 'Plato's Divided Line', *Ancient Philosophy* 16 (1996), pp. 25–46.

[13] N. Smith (1996) notes, 'we are told [by Plato] that the ratios set up represent comparative clarity and obscurity (*saphaneia kai hasatheia*)' (p. 27). This should raise the ears of those familiar with Descartes's terminology of 'clarity' and 'obscurity'.

[14] N. Smith (1996) points out that 'clarity is the measure of the relevant mental states, and truth is the measure of the relevant objects' (p. 27).

Beginning with the bottom most section (shortest in length), Socrates assigns it the role of standing for images. 'By images,' he says, 'I mean, first, shadows, and then reflections in water and on surfaces of dense, smooth, and bright texture, and every thing of that kind . . .' and moving to the section directly above it, 'As the second section assume that of which this is a likeness or an image, that is, the animals about us and all plants and the whole class of objects made by man.'[15] Thus, the shortest section stands for images of things and the proportionately longer section, immediately above it, stands for the 'originals' of the images—that is, the things of which the former are images.

Consider Socrates, then, standing before a mirror. There are two things in play, I think, on which Plato wants us to focus: Socrates and the image of Socrates (the latter located on the mirror's surface). Socrates can be taken to exist independently of the mirror, and vice versa. But, the image of Socrates cannot. It depends for its existence on both Socrates and the mirror. Even so, Plato focuses on the relation between Socrates and the image. The image is said to be 'of' Socrates, not of the mirror—this, even though the image is located on the mirror.[16] It is said to be 'of' Socrates because the image *represents* Socrates. Plato does not say how it is that the image comes to have the representational relation it does to Socrates. But, in highlighting representation, Plato seems to be emphasizing a special kind of ontological dependence: the image of Socrates depends for its being *what* it is (an image *of Socrates*) on the being of Socrates, in a way that Socrates does not depend for his being what he is on the being of the image. To be sure, if the mirror were destroyed, this image would also be destroyed. So, there is a kind of dependence that needs to be worked out between the image and the mirror—but this is something that Plato does not do within the context of his discussion of the analogy.

Plato's concern focuses on the fact that images have a special relation to their originals. In light of the example just introduced, the discussion focuses on the fact that this image has a special relation to Socrates, who is the 'original' of the image, and is the object that this image is 'of'. As some have argued, the asymmetry in terms of ontological dependence is what determines the corresponding inequality of magnitude in their respective line subsections.[17] Generally speaking, the images are less true, or less real, than the objects they represent—their originals. This is captured in the analogy by the magnitude of line section that stands for images being shorter in length than the magnitude that stands for objects (which are the originals of the images). And so, we assign the following to the two subsections that constitute the Visible section of the line:

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[15] *Republic*, p. 745 (510a).

[16] Casting the image as being 'on' the mirror is my choice. Plato talks about images being 'on' shiny surfaces. At least this is the way Shorey and others translate the text.

[17] Here, I focus on N. Smith's (1996) view.

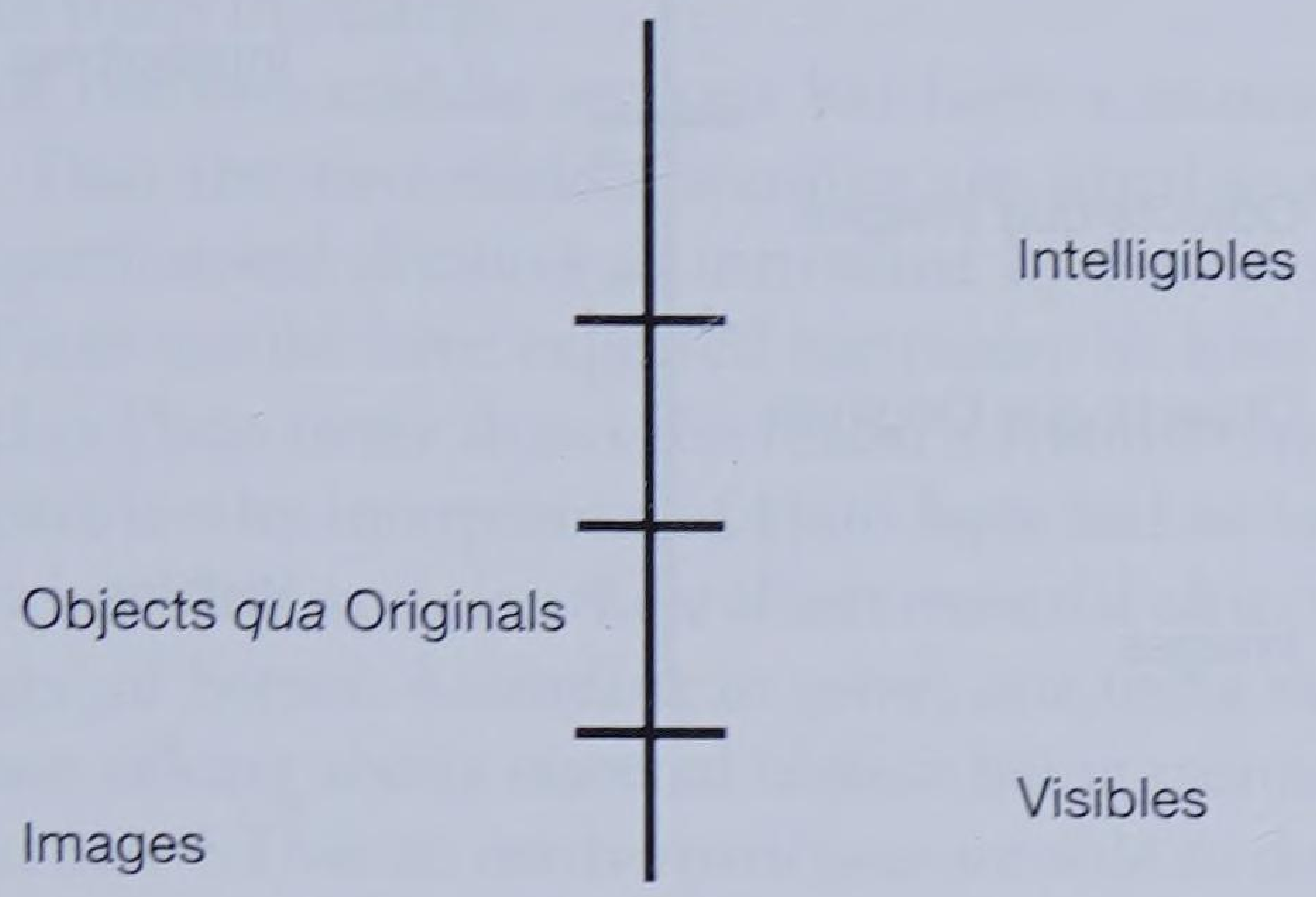

Figure 1.3

After assigning items to the two subsections of the Visible section, Socrates moves to the Intelligible section and to the two subsections constituting it. 'Consider then,' he says, ' . . . the way in which we are to make the division of the intelligible section . . . there is one section of it which the soul is compelled to investigate by treating as images the things imitated in the former division.'[18] It is important to understand what this means. What Socrates is saying, I think, is that the bottom most subsection of the Intelligible section represents those objects just categorized in the uppermost subsection of the Visible section, the things that were understood in the Visible section to be the 'originals' of images. In considering them as items belonging to the Intelligible section, however, they are not taken as they were in the previous Visible section, that is, as 'originals', but now they are taken as themselves being 'images' of something else.[19] It should come as no surprise to those familiar with Plato's work that the 'something else' here will turn out to be the Forms. In line with the Visible section, the image-talk associated with the Intelligible section could also be taken in terms of ontological dependence. Visible objects (read here, physical or material objects) depend for their being what they are on the Forms in a way that the Forms do not depend for their being what they are on the Visibles. The line looks thus:

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[18] *Republic*, p. 745 (510b).

[19] This is a matter of contention among scholars. N. Smith (1996) argues for this way of taking Socrates.

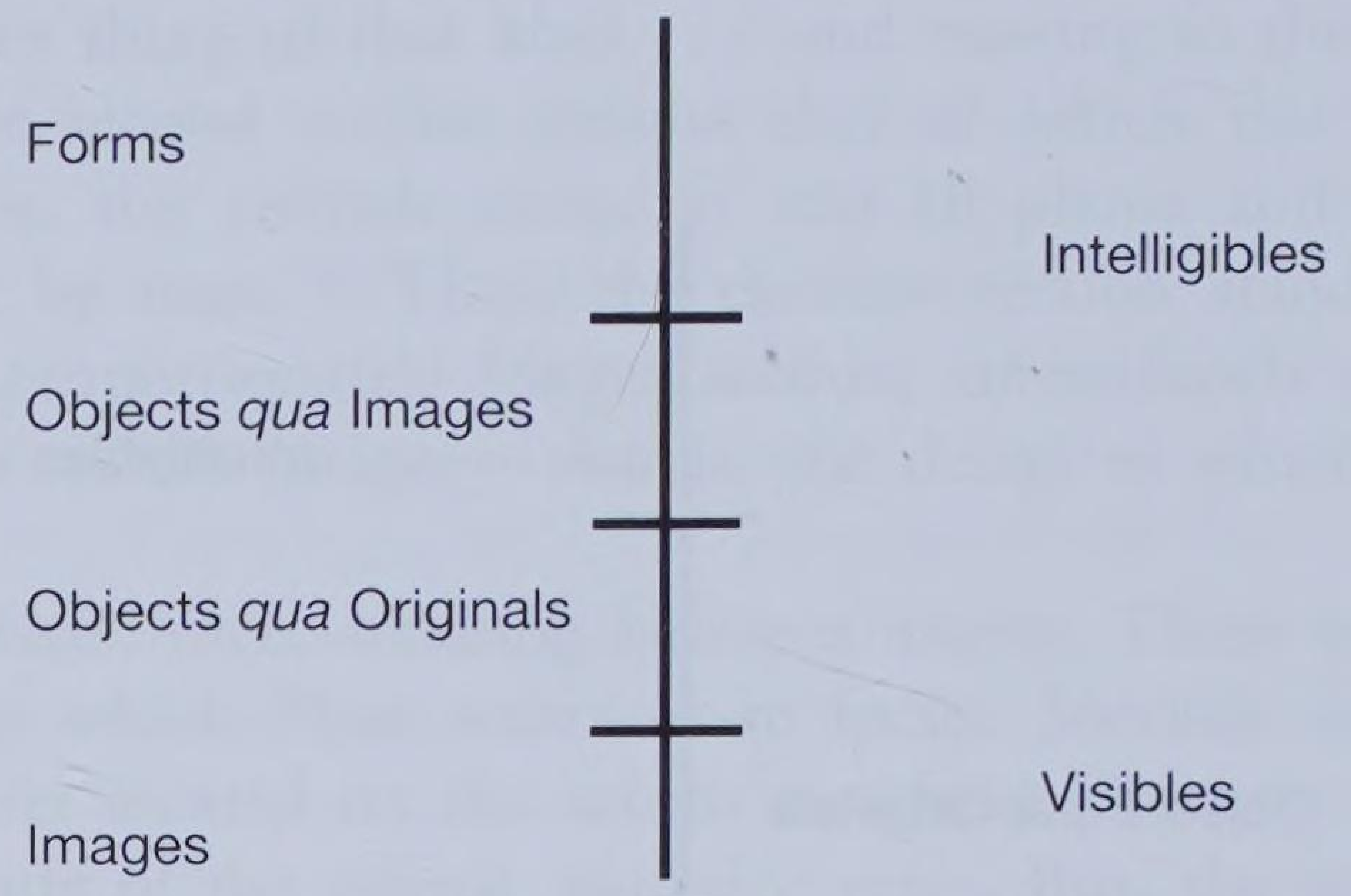
Forms
Intelligibles
Objects *qua* Images
Objects *qua* Originals
Visibles
Images

Figure 1.4

As mentioned above, to each subsection of the line Plato also assigns a kind of mental state. Let me now state them, locating each on the line. To the lowest subsection of the Visible (images) he assigns *eikasia* (imaging or illusion);[20] to the highest subsection of the Visible (objects qua originals) he assigns *pistis* (common sense); to the lowest subsection of the Intelligible (objects qua images) he assigns *dianoia* (thinking); to the highest subsection (Forms) he assigns *noesis* (understanding). And so:

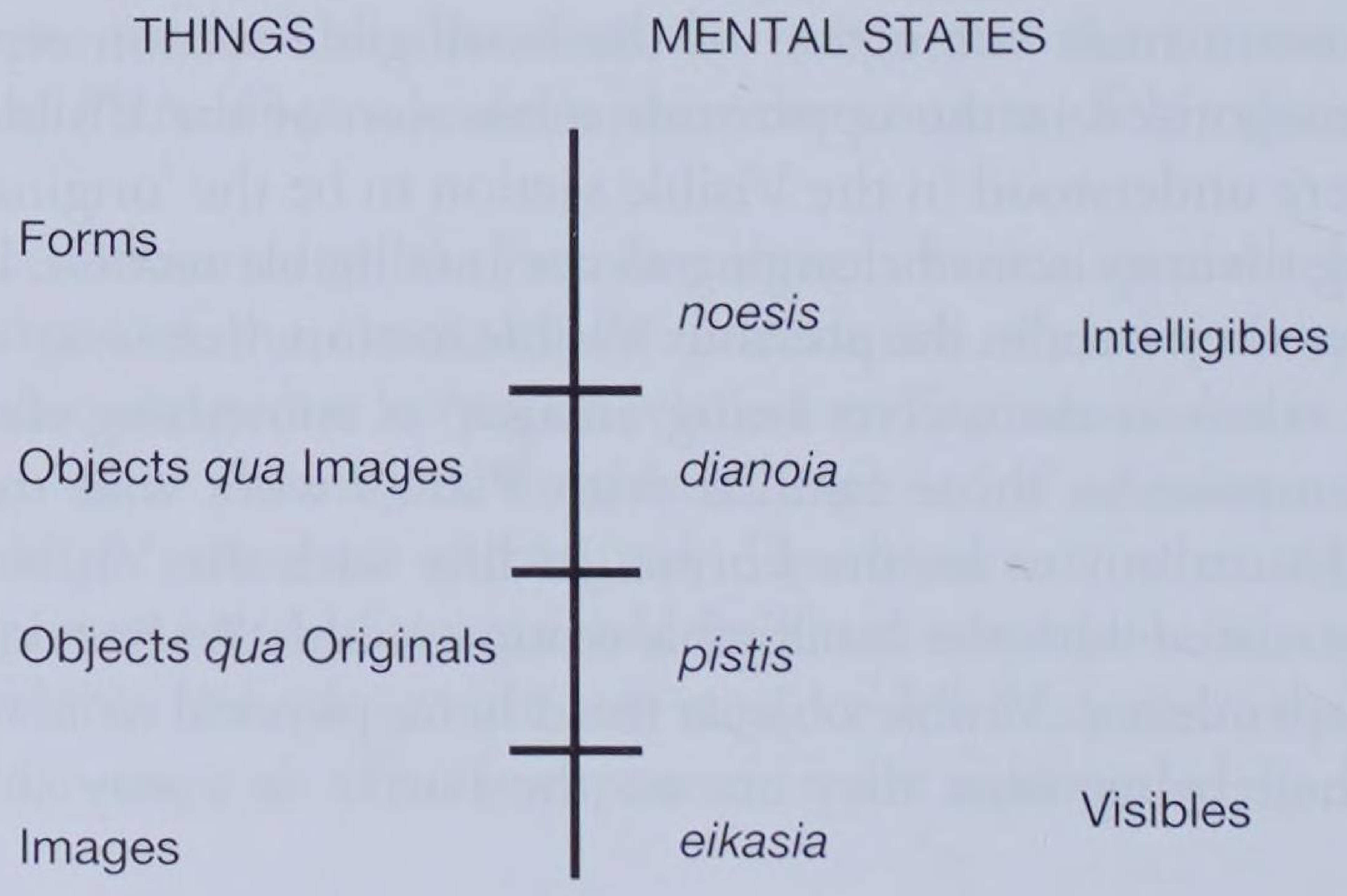
THINGS
MENTAL STATES
Forms
noesis
Intelligibles
Objects *qua* Images
dianoia
Objects *qua* Originals
pistis
Images
eikasia
Visibles

Figure 1.5

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[20] This is N. Smith's interpretation of the four terms—*eikasia*, *pistis*, *dianoia*, *noesis*.

Nicolas Smith notes that if *dianoia* is supposed to be clearer than *pistis* (and that objects qua images are more true or real than objects qua originals), we will have to deal with the fact that their respective magnitudes in length, as presented in the analogy, are equal. For, as it stands, in being equal, the analogy appears to suggest that the two mental states are equal in clarity, and that the two classes of object are equal in truth or reality.

The equality of the two middle sections has been a source of much debate among scholars. That the two middle sections are equal is a consequence of doing the basic geometrical division as instructed by Socrates. And so, this is something that Plato would have expected his reader to have determined.[21] It is worth noting that Plato never draws the reader's attention to this geometrical fact,[22] which in part is why interpreters of Plato have had to look outside of the discussion to see what Plato says elsewhere about material objects being treated or regarded as 'images' of Forms. According to some, one thing that Plato seems to have in mind when talking about material objects being treated as images is the doing of mathematics.[23] That is, mathematicians are able to do what they do by way of regarding material objects in a certain way, where instead of thinking of the box as a material object, the mathematician regards it as a cube, and as such, is able to do something with it that he or she cannot do when thinking of the box as simply a material object. Smith notes that even though Plato takes the material objects of the Visible realm (objects qua originals) and recasts them in light of the Intelligible realm as images (objects qua images), which would suggest that they are simply the same objects though *regarded* differently, Plato must have in mind a sense in which the two *are* different—a sense that is connected to other things he says in the *Republic* and elsewhere.[24] Otherwise, as scholars have pointed out, the analogy fails miserably in making out a difference between *opinion*, which is associated with the Visible, and *knowledge*, which is associated with the Intelligible.[25]

As just mentioned, some have taken Plato to hold that objects qua images are really *mathematical* objects, which, they go on to argue, act as 'intermediaries' between material objects (objects qua originals) and the Forms.[26] Such interpretations do not obviously do away with the problem, however—for, the equality of the middle sections still suggests that *pistis* and *dianoia* are *equally* clear, and that visible (that is, material) objects and intelligible (in this case, mathematical) objects are *equally* true or real. According to Smith, the problem can be addressed if we understand the import of our *regarding* the objects one way or another: 'The same objects can be regarded as empirical originals and as images of intelligibles . . . '[27] where the regarding determines an important difference in both the mental states involved and the objects regarded. For instance, concerning mathematics, he says, 'mathematical study will employ objects that might be regarded as the originals of the sensible realm, but when *used* by the mathematician, these objects will be seen as images of higher realities'.[28] Specifically, they will be considered to be images within the context of the Intelligible realm. So, in what sense is *dianoia*, which arises as a result of dealing in the right way with objects qua images of the Intelligible realm, more clear than *pistis*, which arises as a result of dealing with objects qua originals of the Visible realm?

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[21] N. Smith (1996), p. 42. [22] Ibid., pp. 40–41.

[23] Nolan deals with the connection between imagination and mathematics understood within the context of an analysis of Plato's divided line. Lawrence Nolan, 'The Role of the Imagination in Rationalist Philosophies of Mathematics', in *A Companion to Rationalism*, edited by Alan Nelson (Oxford: Blackwell Publishing, 2005), pp. 224–49. I follow those who see the middle sections of the line as being equal in length.

[24] N. Smith (1996), p. 31.

[25] See A. J. Boyle, 'Plato's Divided Line: Essay I: The Problem of *Dianoia*', *Apeiron* (1973) 7: pp. 1–11, and 'Plato's Divided Line: Essay II: Mathematics and Dialectic', *Apeiron* (1974) 8: pp. 7–21; and R. C. Cross and A. D. Woozley, *Plato's Republic: A Philosophical Commentary* (New York: St. Martin's Press, 1966).

[26] N. Smith (1996), p. 35.

Smith contends:

> The objects at each level were given by Plato (and were taken by their consumers) under different descriptions. So, one at the level of *pistis* would be associated with the visibles *qua* visible; one at the level of *dianoia* would be associated with the visibles *qua* intelligible images. Their different attitudes towards such objects would produce different mental states, of different degrees of clarity. But their differing attitudes would also produce a substantial difference in their perception of the truth of these objects . . . What is relevant, then, is not the *ontological identity* of these objects, as some have argued, but, rather, the way in which such objects are conceived at each of the relevant subsections.[29]

Regarded or conceived in such radically different terms, and 'employed under completely different, and incompatible descriptions,' then, 'it is almost as if there are two different sorts of objects at these levels'.[30] I would like to pause for a moment and make Smith's point clearer.

As a boy, rummaging through a closet, I ran across an old chess set. When I opened the box I was delighted to discover a black and white checkered board and a bag full of small wooden objects. Of course, at the time I had no idea that they were game pieces—in fact, I had no idea that this was a game (though I had my suspicions). I played with the pieces, especially the ones that looked like castles and horses. I eventually brought the set to my father, who was more than eager to show me how to play. It was tough going, to say the least. But, I learned. The pieces took on a very different 'look' to me after that, for I now *understood* the piece that looked like a horse, for example, to be a knight, where my understanding of a knight in this context was informed by the 'move' that it was allowed to make when on the board. There was nothing in the piece itself—that is, the piece qua carved hunk of wood—that possessed the 'move'. Rather, this was something brought to the piece by way of my understanding the piece as determined by the game. It was brought to light by my *regarding* the object a certain way.

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[27] N. Smith (1996), p. 38.

[28] Ibid., emphasis mine.

[29] Ibid., pp. 41–2.

[30] Ibid., p. 41.

The difference between the castles and horses changed significantly once I understood them under the description of the game—as rooks and knights. To be sure, there does not appear to be anything inherently clearer in my understanding the piece as a knight as opposed to simply a carved piece of wood. That is, taken in isolation from all else, one does not appear intrinsically clearer to me than the other. Rather, the clarity, or my being *clearer*, is to be measured by my now regarding the piece in a certain way, in light of its role *in the game*. Clarity is a teleological concept. Once I regard this wooden object *as* a knight, there is something that I can do that was not *possible* for me before regarding it as such.[31]

Likewise, I think, Smith is arguing that in regarding certain visible (sensible) objects *as* images of Forms, which includes regarding them simply as *shaped*, for instance, certain possibilities open up, one of which is the possibility of doing geometry; another (more importantly) is the acquiring of knowledge. Simply regarding visible objects *as visibles* is not enough for doing geometry and acquiring knowledge, in the same way that regarding the objects as hunks of carved wood is not enough for being able to use them to play the game of chess. Thus, there seems to be a sense in which *dianoia*, the mental state associated with regarding objects as images, is clearer than *pistis*, the mental state associated with regarding objects as originals—*dianoia* allows me to do something that *pistis* does not, specifically it allows me to acquire knowledge. Another way to put this is that, although the line subsections are the same length, when put in the context of the larger subsections in which each belongs (*pistis* in the Visible, *dianoia* in the Intelligible), *dianoia* can yield real knowledge (*noesis*), whereas *pistis* cannot. There is more available to someone who is in a state of *dianoia*—which is the sense in which it is a superior mental state to *pistis*. This, I take it, is Smith's view.

Aristotle seems to have adopted some of this, though he clearly would disagree with Plato (or the Platonist) if one took the objects qua images to be extra-mental or mind-independent. For Aristotle they are simply the result of abstraction. Jonathan Lear's account begins by showing how Aristotle introduces the process of abstraction in the *Physics*:

> The next point to consider is how the mathematician differs from the physicist. Obviously physical bodies contain surfaces, volumes, lines, and points, and these are the subject matter of mathematics . . . Now the mathematician, though he too treats of these things

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[31] Here I simply mean that it was not possible for me *as an agent*—clearly it was always *logically* possible. I could have used another example—for instance, in the same way that in my not knowing the English alphabet, it will not be possible for me to read written English.

> (viz., surfaces, volumes, lengths, and points), does not treat them *as* the limits of a physical body; nor does he consider the attributes indicated as the attributes *of* such bodies. That is why he separates them, for in thought they are separable from motion (*kinesis*), and it makes no difference nor does any falsity result if they are separated. Those who believe the theory of the forms do the same, though they are not aware of it . . . Similar evidence is supplied by the more physical branches of mathematics, such as optics, harmonics, and astronomy. These are in a way the converse of geometry. While geometry investigates physical lengths, but not *as* physical, optics investigates mathematical lengths, but *as* physical, not *as* mathematical.[32]

Abstraction is characterized as separating in thought something from something else. In the case of mathematics, the mathematician separates in thought, for example, the box's shape (cube) from the physical box, and subsequently deals with the box *as* a cube. Note that Aristotle suggests that if one thinks that what they have post-abstraction is a Platonic Form, one simply is unaware that the thing they have before the mind (the cube) is a mental entity got by the mental activity of abstraction. Lear notes a second mistake that the Platonist makes, namely, 'they choose the wrong things to separate'.[33] For example, they try to separate from matter things that are unintelligible apart from being enmattered.[34] Aristotle's paradigm case is that of *snub nose*. The 'account of *snubness* cannot merely specify shape; *snub* must be a shape embodied in a nose.'[35]

As we will see later, we find both Platonic and Aristotelian trends in rationalists like Descartes. For instance, in the Third Meditation, Descartes draws an important distinction between two ideas of the sun.[36] The first is his sensory idea, which exhibits the sun as being very small (and one could add here yellow, hot, flat, and so on). The second is his intellectual idea of the sun, which casts the sun as a colourless sphere that is much greater in size than the earth. This idea is the one that is useful in doing mathematical astronomy, and, he says, better represents the sun (insofar as it better resembles it).[37] Other cases of abstraction support this reading of Descartes. For example, Descartes says that one can abstract motion or shape from a body (an extended thing), which allows one to deal mathematically with that body, but for all that, these modes (motion and shape) cannot be conceived in complete ideational isolation from body—they cannot be excluded from it.[38] In drawing the distinction between the *seen*-sun and *intellected*-sun, Descartes appears to be aligned with the distinction that

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[32] This comes from *Physics* B2, 193b 23—194a 12. I use Jonathan Lear's translation, quoted from his article 'Aristotle's Philosophy of Mathematics', *The Philosophical Review,* XCI, no. 2 (April 1982), p 162.

[33] Ibid., p. 163.

[34] Lear (1982).

[35] Ibid., p. 163. Lear refers to Aristotle's discussions of *snub nose* in *Metaphysics* 1025b 31, 1030b 29, 1035a 26, 1064a 23, 1030b 17, 1035a 5, 1064a 25.

[36] AT VII 39; CSM II 27.

[37] Ibid.

[38] AT VII 120–21; CSM II 86.

Plato draws between visibles and intelligibles, and more specifically between *pistis* and *dianoia*.[39] Yet, like Aristotle, he recognizes that in cases of abstraction it would be a mistake to think that what one has before the mind post-abstraction is something that exists independently of matter *or* thought.[40] For, along the lines of Aristotle, Descartes characterizes abstraction as separating in thought a mode (or quality) from other modes, or a mode from the substance of which it is a mode.[41] And, he says, it would be an error to think that shape, for instance, though abstracted from the thing (corporeal substance) could exist independently of that thing, and what is more, independently of the very thought that one is having at the moment.[42]

The mixture of both Platonic and Aristotelian positions appears again in the Fifth Meditation, when he introduces the notion of a true and immutable nature.[43] The example he explores is that of the geometrical figure, the triangle. He says both that the true and immutable nature of the triangle does not exist 'outside' of thought (an Aristotelian position), and at the same time that the nature is such that since it was not produced by his thought, it maintains a unique kind of mental independence (a Platonic position).[44] But, we will focus on this more carefully when we get to Descartes in Part II.

It is worth noting that, although there are four line sections (and hence four categories of things) for Plato, there really are only *three* sorts of item being considered: images, objects, and the Forms. Objects are categorized twice: once with respect to the Visible section, and a second time with respect to the Intelligible section. And so, Plato is in essence relating objects first with images (of which they are said to be the originals), and second with the Forms (of which they are said to be images).

The *objects*, as listed by Socrates, are such things as animals, plants, artefacts of all kinds, and so on. As was suggested above, objects in this context are what are ordinarily taken (pre-philosophically) to be 'physical' or 'material' things. The mental state associated with such objects is *pistis*, or common sense. Clearly, when they are taken with respect to the Forms, we are taking them in a way that is not so ordinary. Taking them in this way is in the context of a philosophical investigation. That is, it is upon philosophical reflection that what is ordinary in one sense (understood in terms of the Visible or Sensible) is extraordinary in another (understood in terms of the Intelligible). The mental state associated with such objects (taken in this extraordinary sense) is *dianoia*, or thinking.

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[39] I should add, however, that unlike Plato (or Aristotle), Descartes does not ultimately recognize essences or natures of 'natural' kinds. So, for Descartes there is no innate idea of *the sun* that would serve as a counterpart to Plato's Form. Even so, Descartes does appear to recognize the distinction between sensed objects and 'intellected' (or purely intellectual) objects.

[40] AT VIIIA 31; CSM I 215–16.

[41] See, for instance, AT VII 120; CSM II 85–6, AT VII 220; CSM II 155, AT X 413; CSM I 41, AT X 458; CSM I 69, AT III 421; CSMK III 188, AT III 474–4; CSMK III 201–2, AT VIIIA 31; CSM I 215–16. Technically speaking, Descartes seems to prefer attribute talk over mode talk when it comes to abstraction. I will address this technical point later in the book.

[42] AT VIII 31; CSM I 215–16.

[43] AT VII 64; CSM II 44–5.

[44] Ibid.

The objects, even though introduced in the analogy after images are introduced, seem to be the salient *starting* point of two kinds of investigation, which Socrates will only briefly describe in his discussion of the divided line. This starting point is confirmed when Socrates restates the import of the analogy to Glaucon:

> By the distinction that there is one section of it which the soul is compelled to investigate by treating as images the things imitated in the former division, and by means of assumptions from which it proceeds not up to a first principle but down to a conclusion, while there is another section in which it advances from its assumption to a beginning or principle that transcends assumption, and in which it makes no use of the images employed by the other section, relying on ideas only and progressing systematically through ideas.[45]

Starting with objects, then, there are two sorts of investigation to be made. First, there is the sort of investigation that would start with objects, in the above passage referred to as 'assumptions' (*hypothasus* is the word Socrates uses), the things assumed or hypothesized, and would proceed 'downwards' to a discovery of their 'conclusions', which in this case are the images. This sounds a bit strange to us, of course, but there are ways of dispelling some of this. For instance, using causal language for a moment,[46] the idea seems to be that with respect to this sort of investigation, one begins with what we call a *cause* and derives from it its *effect*. Effects will depend on their causes as images will depend on objects (qua originals). Alternatively, using argumentative language, *conclusions* will depend on their *assumptions* as images (or effects) will depend on objects (or causes). The object-image, cause-effect, and assumption-conclusion ways of talking are analogous (their grammatical structures are isomorphic), and capture an important form of dependence (presumably a form of *necessary* dependence). The sort of investigation that moves downward from object to image (or cause to effect, or assumption to conclusion), as we will soon learn, is what the ancients referred to as *synthesis*.

By contrast, the other sort of investigation begins with objects, but this time attempts to get at a first principle. This will be like beginning with an effect and trying to discover its cause (or causes), or beginning with a conclusion and trying to discover its assumption (or assumptions). This second sort of investigation moves 'upwards', where Socrates casts the move as one that 'transcends' the assumption (or object). The ancients referred to this sort of investigation as *analysis*. As was suggested earlier, to perform this analysis we must regard the objects in a certain way, a way that will allow us to *do* something with them—here the 'doing' is our finding how they are related to other things, where the web of objects and their relations are understood to form a well-ordered system, everything ultimately rooted in what was called *first principles*. But, I am getting ahead of myself. Let's pause and get a sense of the kind of investigation called *analysis*.

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[45] *Republic*, p. 745 (510b).

[46] Causal language is employed by Aquinas, for instance, to explicate analysis and synthesis. And so, my introducing it is not unique.

2

Plato's Early and Late Methods

According to Toulmin and Goodfield, 'for Plato, the logical backbone of science was to be provided by geometry: for Aristotle, by classification'.[1] Although geometry struck both Plato and Aristotle as a model for science, Toulmin and Goodfield fail to see the significance of Plato's focus on classification, found in his discussions of analysis. Kenneth Sayre argues that Plato first introduces the method of analysis in the *Theaetetus*.[2] Norman Gulley claims that Plato first introduces the method earlier in the *Meno*.[3] Even so, like most scholars who realize the significance of analysis for Plato (as opposed to geometry proper), they agree in their characterizing the method as one of collection and division, or classification. In the later dialogues the method of analysis replaces what Sayre refers to as the method of hypothesis, found in the early dialogues. The details of the new method are subsequently worked out in the *Phaedrus* and the *Sophist*.

The earlier method of hypothesis was a method that helped Socrates determine the *consistency* of claims held by his interlocutors.[4] His interlocutor would offer a definition of *A*, say, and then the method would be employed in determining whether other beliefs crucial to the interlocutor's view were consistent with *A*. Euthyphro, for example, after some prodding by Socrates, defines 'piety' as that which the gods love. That which the gods hate, he goes on to say, is the definition of 'impiety'.[5] Three other beliefs held by Euthyphro are brought to light by Socrates:

1) The gods can disagree with one another,[6]
2) The pious and impious are opposites,[7]

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[1] Stephen Toulmin and June Goodfield, *The Architecture of Matter* (New York: Harper & Row, 1962), p. 75.

[2] Kenneth M. Sayre, *Plato's Analytic Method* (Chicago: University of Chicago Press, 1969).

[3] Norman Gulley, *Plato's Theory of Knowledge* (London: Methuen & Co. Ltd, 1962), pp. 14–15, at 86e ff. It should be noted that Sayre also finds the method in the *Meno*, but argues that it is not fully developed until the *Theaetetus*, Sayre (1969), p. 57.

[4] Not everyone holds this. Sayre (1996) thinks that this view is problematic (Sayre, p. 52). The sort of view as I lay it out here is defended by R. Robinson in *Plato's Earlier Dialectic* (Oxford: Clarendon Press, 1953).

[5] *Euthyphro*, in *Plato: The Collected Dialogues*, translated by Lane Cooper, edited by Edith Hamilton and Huntington Cairns (Princeton: Princeton University Press, 1989), pp. 169–85. Euthyphro makes the definition at 6e, (Hamilton and Cairns, p. 174).

[6] *Euthyphro*, 6b and 7b (Hamilton and Cairns, pp. 174–5).

[7] *Euthyphro*, 7a (Hamilton and Cairns, p. 175).

3) Whatever the gods believe is just, they love; whatever they believe is unjust, they hate.[8]

In light of (1), Socrates supposes a case in which some gods believe some object *f* to be just, while other gods disagree and believe *f* to be unjust. Euthyphro allows the supposition. This supposition and (3) yield:

4) Some gods love *f* and some gods hate *f*.[9]

The definitions of *piety*, *impiety*, and (4) in turn yield:

5) *f* is pious and *f* is impious.[10]

Of course, (2) asserts that the pious and the impious are opposites, which here entails that the same thing cannot be both at the same time in the same respect. So, *f*'s being pious on Monday and impious on Tuesday is possible (assuming that the gods can change their minds), but *f* cannot be *both* pious *and* impious *at the same time*. And so, given that Socrates is taking the above to be expressing that *f* is both pious and impious at the same time, (5) appears to be a contradiction.

The argument that Socrates constructs is a *reductio ad absurdum*. What Socrates has shown is that at least one of Euthyphro's beliefs connected to his view on piety is false. In today's terms, given that a valid argument with a false conclusion must have at least one false premise, and that the above argument is valid with a false conclusion (a contradiction in this case), it follows that at least one of its premises is false. The set of premises, together with Euthyphro's definition of *piety*, is inconsistent. This method, I think, is what Sayre is referring to by the phrase 'method of hypothesis'.[11] The method puts forward an hypothesis, which in the above case is Euthyphro's definition of piety, and then works towards an argument whose conclusion is a contradiction. Assuming that the argument is valid, such an argument would show that at least one premise is false, in which case at least one claim crucial to the interlocutor's position is inconsistent with other claims crucial to that position—most notably, it is inconsistent with that which was hypothesized (which in the above case is the definition of *piety*). The method usually allowed the interlocutor to abandon one of the crucial claims, or, if the interlocutor wished to maintain all of the crucial claims, to abandon the hypothesis. A new hypothesis would be offered, and the investigation would continue forward.

Francis Cornford at one point had argued that 'Definitions are not hypotheses: they do not assert the existence of anything, but only state a meaning that must be understood. Hypotheses [on the other hand] are assumptions of the existence of things defined.'[12] So, if Cornford is right, the above characterization of the early method is wrong—for the definition of *piety* will not count as an hypothesis. Three years after making the above assertion, however, though continuing in his emphasis on meaning and its connection to definition, Cornford said about the method: 'A tentative suggestion ('*hypothesis*') put forward by one speaker is corrected and improved until the full *meaning* is clearly stated. The criticism that follows may end in complete rejection or lead on to another suggestion' which, if examination has been skilfully conducted, he said, will lead investigators 'nearer the truth'.[13] This later view, it seems to me, allows definitions to count as hypotheses. And so, in the end, the views of Cornford and Sayre appear compatible with how the method was characterized above.

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[8] *Euthyphro*, 7e (Hamilton and Cairns, p. 176).

[9] *Euthyphro*, 8a (Hamilton and Cairns, p. 176).

[10] Ibid.

[11] Socrates also uses *argument by analogy* to show inconsistency, by I will not discuss it here.

[12] F. M. Cornford. 'Mathematics and Dialectic in the Republic VI.–VII.(I.)', *Mind* (New Series), vol. 41, no. 161, (Jan. 1932), p. 41.

To be sure, today's philosopher would not hold that, in asserting a definition, he or she would in turn be assuming the existence of anything. I am not sure that even in the case of understanding Plato's method of hypothesis that anything really turns on *insisting* on the existence of the thing being defined. Be that as it may, when referring to piety, Plato's own phrasing is '*ta hosion auta*', which is 'the pious itself'. If the definition *were* true, it would express the nature or form of piety—the pious itself—which *is* an eternal being, a genuine substance of the cosmos. Here, perhaps borrowed from his predecessor Parmenides, *truth* is connected to *being*, such that if there is the one, there is the other. And so, at least conditionally, were the definition true it would entail the existence of the thing defined. As an aside (we will discuss this later), we find this connection affirmed even by Descartes. In the Fifth Meditation, for instance, he says, 'for it is obvious that whatever is true is something'.[14] This sort of connection (between truth and being) not only seems to be the case for Plato (or later for Descartes), but for Aristotle. And, as commentators have pointed out,[15] Aristotle's discussion of definition in *Posterior Analytics* suggests that a definition asserts of an *existent* thing what it is.[16] Even so, we can understand the method of hypothesis, or at least how it worked, regardless of where Plato stood with respect to this point.[17]

As stated earlier, analysis is a method of *collection and division*. Unlike the method of hypothesis, the method of analysis is not a method specifically used for determining consistency. But it is also not exactly the ancient *geometrical* method that went by the same name—something that Toulmin and Goodfield failed to see. Both Sayre and Cornford agree that a strict rendering of the method as being the one belonging to ancient geometry does not align with what the texts actually reveal. This has not only been a source of confusion, as seen with

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[13] Ibid., p. 30, emphasis mine.

[14] AT VII 65; CSM II 45.

[15] For more on this, see Robin Smith's 'Aristotle's Logic', *The Stanford Encyclopedia of Philosophy*, Edward Zalta (ed.), (Fall 2004) sections 7.1, 7.5.

[16] Aristotle. *Posterior Analytics*, translated by Jonathan Barnes (Oxford: Clarendon Press, 1994), Book B, Chs 3–10, 90b–94a, pp. 54–65.

[17] R. Smith (2004) points out that although Aristotle recognized definitions that do not assume the existence of the things they define, he does not appear to have been interested in them. Smith, 'Aristotle's Logic' (2004), section 7.1.

Toulmin and Goodfield, but has been a source of serious contention among their scholarly predecessors and contemporaries.[18]

Richard Robinson, for instance, argues for a strict geometrical rendering of analysis.[19] He reconsiders *the* ancient source for such a rendering: a description of analysis and synthesis found in Pappus' *Mathematicae Collectiones* (written between 325 and 340 CE). Pappus writes:

> Analysis then takes that which is sought as if it were admitted and passes from it through its successive consequences (*dia ton egthes akolouthon*) to something which is admitted as the result of synthesis: for in analysis we assume that which is sought as if it were (already) done (*gegonos*), and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by so retracting our steps we come upon something already known or belonging to the class of first principles, and such a method we call analysis as being solution backwards (*anapalin lusin*).
>
> But in synthesis, reversing the process, we take as already done that which was last arrived at in the analysis and, by arranging in their natural order as consequences what were before antecedents, and successively connecting them one with another, we arrive finally at the construction of what was sought; and this we call synthesis.
>
> Now analysis is of two kinds, the one directed to searching for the truth and called *theoretical*, the other directed to finding what we are told to find and called *problematical*. (1) In the *theoretical* kind we assume what is sought as if it were existent and true, after which we pass through its successive consequences (*ton hegses akalouthon*), as if they too were true and established by virtue of our hypothesis, to something admitted: then (a), if that something admitted is true, that which is sought will also be true and the proof will correspond in the reverse order to the analysis, but (b) if we come upon something admittedly false, that which is sought will also be false. (2) In the *problematical* kind we assume that which is propounded as if it were known, after which we pass through its successive consequences (*ton heses akalouthon*), taking them as true, up to something admitted: if then (a) what is admitted is possible and obtainable, that is, what mathematicians call *given*, what was originally proposed will also be possible, but if (b) we come upon something admittedly impossible, the problem will also be impossible.[20]

I shall return to this passage in Part II, and will argue that Plato's (but more importantly for me, Descartes's) method is akin to what Pappus refers to as *problematical* analysis—where P is given and the investigation attempts to uncover the conditions that make P possible.[21] For now, however, I shall continue with an examination of Robinson's reading of the passage.

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[18] Cornford (1932), p. 48.

[19] Robinson (1936).

[20] I quote from Robinson (1936), p. 466, (see Euclid), who uses the translation of Thomas Heath, in his *The Thirteen Books of Euclid's Elements*, 2 vols, (New York: Dover Publications, 1956). The passage appears in vol. I, p. 138.

[21] Stephen Gaukroger also identifies this kind of analysis as the one Descartes took up, though my reasons for identifying it as the method Descartes chooses are different from Gaukroger's. For Gaukroger's study, see 'The Nature of Abstract Reasoning: Philosophical Aspects of Descartes' Work in Algebra', in *The Cambridge Companion To Descartes*, John Cottingham (ed.) (Cambridge: Cambridge University Press, 1992), p. 106.

According to Robinson, analysis 'was a method of discovery, a method of discovering either the proofs of geometrical propositions or the solutions of geometrical problems',[22] which, he says, 'was followed by a synthesis, and the latter constituted in the first place a check on the analysis, to make sure that there had been no error . . . [and in the second place] constituted the actual proof or solution for the sake of which the analysis was undertaken'.[23]

If one assumed or hypothesized, say, geometrical proposition P, analysis unfolds in our working 'backwards' or 'upwards' in our attempt at hooking up P with some first principle (or class of first principles). As we saw above, this is along the lines of Plato's remarks in Book VI of the *Republic*. Even so, Robinson casts the investigation *solely* in terms of justification (something to which Plato does not limit himself).[24] To better understand Robinson's view, let's say that we discover that Q underwrites P in the sense that if Q were true P could not be false. We might say, then, that P is derivable from Q. This, of course, does not guarantee the truth of Q. But, suppose that we already know that Q is derivable from some first principle R. Given that R is necessarily true (it is a first principle after all), and Q is derivable from R (keeping to the idea that by 'derivable' here we mean that if R is true Q cannot be false), the truth of Q has already been established. Now, since P is derivable from Q, and Q is true, P is true. Analysis is our working from P back to R, R being a first principle. Synthesis, then, is our working the other way—that is, 'forwards' or 'downwards'. Beginning with R, deriving Q, and then deriving P. Thus, synthesis simply is the *proof* from R to P.

Insisting on how analysis and synthesis were employed by the ancient geometers, Robinson argues that for the above to work, 'the implications must be reciprocal. Not merely must (1) imply (2), but (2) must imply (1).'[25] Thus, in terms of the above case, not only does R imply Q, but Q implies R; and not only does Q imply P, but P implies Q. So, instead of '$R \Rightarrow Q$' and '$Q \Rightarrow P$', where '$\Rightarrow$' denotes implication (so that '$R \Rightarrow Q$' is read 'R implies Q'), or in terms of material implication, the conditionals '$R \rightarrow Q$' and '$Q \rightarrow P$' (it is not clear which logical machinery Robinson has in mind), Robinson seems to claim that analysis and synthesis establish '$R \Leftrightarrow Q$' and '$Q \Leftrightarrow P$', or the biconditionals '$R \equiv Q$' and '$Q \equiv P$'. Either way, there seem to be several problems lurking here.

One problem is, if what is established is reciprocal implication (or in terms of material implication, the biconditional) between R and Q, then there is no *logical* difference between the relation discovered holding between Q and R when doing analysis, and the relation holding between R and Q when doing synthesis. So, on Robinson’s view, when performing analysis one would be *deducing* the first principle R from Q. This threatens R’s being taken to be a first principle (first principles are not *deducible* from other propositions). The problem is highlighted in Robinson’s use of Euclid’s proof of proposition XIII.1. There, Robinson casts the proof in algebraic terms, where the proof is moved ‘upwards’ or ‘downwards’ (depending on whether he is performing analysis or synthesis) by way of applying algebraic rules. Each line of the proof is an equation, thus taking the form ‘$P = Q$’. But, if ‘$P = Q$’ and ‘$Q = R$’, where R is a first principle, then P and Q are *equivalent* to R. In other words, Robinson’s way of construing what is established by way of analysis and synthesis—namely, that P, Q, and R are equivalent—fails to capture the idea of a *hierarchy* of propositions, where those lower in the hierarchy are derived from those higher in the hierarchy.[26]

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[22] Robinson (1936), p. 464.

[23] Robinson (1936).

[24] I believe that Plato's method of analysis is closer to what Pappus calls *problematical*—that is, the aim of the investigation in hypothesizing P is to discover what conditions underwrite the very *possibility* of P.

[25] Robinson (1936), p. 465.

Without going too far afield, it seems to me that what Robinson should be considering is not reciprocal implication but, rather, the sort of thing one does when trying to produce a proof for a theorem in logic by working backwards. Take, for example, one’s trying to show that $\sim A \rightarrow (A \rightarrow B)$ is a theorem. We might first try our hand at breaking up the conditional, entertaining the possibility that $\sim A \rightarrow (A \rightarrow B)$ was got by way of Conditional Proof (CP). Thus:

1.	$\sim A$	
2.	$A \rightarrow B$	
3.	$\sim A \rightarrow (A \rightarrow B)$	1–2 CP

Now, can we get $A \rightarrow B$ from $\sim A$? Well, by Addition we can add B to $\sim A$ and get $\sim A \vee B$, and then by Implication we can move from $\sim A \vee B$ to $A \rightarrow B$. So:

1.	$\sim A$	
2.	$\sim A \vee B$	1 Add
3.	$A \rightarrow B$	2 Impl
4.	$\sim A \rightarrow (A \rightarrow B)$	1–3 CP

If the sort of thing we did in producing the above proof was analysis (I will argue below that it is not exactly this), it seems yet again that Robinson’s insistence on *reciprocal* implication would be wrong.

Robinson’s emphasizing *implication* brings with it several other problems. For example, as would be a concern for the early moderns (and their medieval and scholastic predecessors), there were at least two interpretations of implication, namely, *intensional* and *extensional*—something that Robinson does not address.

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[26] In light of what was said above about the cause-effect way of casting what is sought in these investigations, the trouble is amplified. For, given the equivalence between P and Q, where Q is the cause and P the effect, what is shown is that P is the cause of itself.

This aside, there is always the concern about anachronism, especially when it comes to discussing historical periods using terms not obviously applicable to those periods. Fortunately, Robison's problems are not anachronistic in nature. N. I. Styazhkin's impressive historical treatment of the development of the concept of implication reveals that the predecessors of the early moderns commanded a sophisticated grasp of both interpretations.[27] The above mentions of *implication*, *reciprocal implication*, *conditional*, and *biconditional*, then, were they properly explained, would not have been completely lost on them. Duns Scotus, for instance, drew a distinction between formal and material implication,[28] and Raymond Lull developed truth tables for the connectives 'and' and 'or' that corresponded 'exactly to the modern rules for truth of conjunctions and disjunctions', even providing analyses of propositions whose operators were 'not' and 'if . . . then'.[29] So, a discussion about such concepts with respect to the early moderns and their predecessors is not obviously anachronistic. Even so, Robinson owes us some explanation of how *he* means *implication* here.

A standard extensional interpretation of implication found in the medieval and scholastic periods, for example, which seems to have been rooted in the work of the ancients, was built around the idea of class membership. Thus, to say, 'All men are mortal' was to express the idea that the class *man* is a member of the class *mortal things*. One way to express this universal statement is as the conditional: if some thing *x* is a man, then that thing *x* is a mortal thing. Thus, supposing that Socrates is a man, and all men are mortal, it will follow that Socrates is mortal.[30] Aristotle considers the anatomy of such a logical move.[31] In the *Categories*, for instance, after drawing a distinction between simple and composite forms of speech, Aristotle analyses what appear to be the various 'parts' of speech into four categories (though some ontology emerges during the analysis), using the two locutions '*said of* (a subject)' and '*in* (a subject)'.[32] The four categories are:

1) things said of but not in a subject;
2) things not said of but in a subject;
3) things both said of and in a subject;
4) things neither said of nor in a subject.[33]

Something B is 'said of' another, A, if and only if, B is predicated of A. Though not perfectly clear, Aristotle seems to have in mind the idea that, at the logical ground floor 'B' denotes a class[34] and 'A' an individual. And so, given that Socrates is a member of the class *man*, *man* is predicated of Socrates—'Socrates is a man.' Note that it is not, 'Socrates is man.' Supplying the indefinite article 'a' here is important (it is expressing class membership), especially when predicating of what Aristotle calls a *primary substance*, which is an individual—for example, an individual man or horse.[35] A primary substance emerges in the above scheme as an item belonging to category (4). For, a primary substance is neither *said of* nor *in* another thing.[36] A secondary substance is the species (or class) to which the individual belongs.[37] Classes belong to category (1) (and arguably to category (3)).

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[27] N. I. Styazhkin, *History of Mathematical Logic From Leibniz to Peano* (Cambridge: MIT Press, 1969).

[28] Ibid., pp. 30–6.

[29] Ibid., pp. 10–13.

[30] To get this exactly right we would have to clean up the language, of course, since strictly speaking he is a member of a class that is a member of the class *mortal things*.

[31] For instance, in Chapter 4, Ib10, Chapter 5, 2a25 and 3a10–17. Aristotle, *Categoriae* in *The Works of Aristotle*, vol. I, edited by W. D. Ross, translated by E. M. Edghill (Oxford: Oxford University Press, 1928).

[32] Aristotle, *Categoriae*.

[33] Ibid., Chapter 2, Ia20–Ib10.

Consider what I think is a remark related to the idea of implication. Aristotle says:

> When one thing is predicated of another, all that which is predicable of the predicate will be predicable also of the subject. Thus, 'man' is predicated of the individual man; but 'animal' is predicated of 'man'; it will, therefore, be predicable of the individual man also: for the individual man is both 'man' and 'animal'.[38]

So, if Socrates is a member of the class *man*, and 'man' is thus predicable of Socrates, and *man* is a member of the class *animal*, then Socrates is a member of the class *animal*, in which case it follows, he says, that 'animal' is predicable of Socrates.

The classes *man* and *animal* are related: the former is *included in* the latter. In part, what this means is that *man* is included in *animal* insofar as *animal* 'covers a larger field'[39] of cases, having a 'wider extension than'[40] *man*. A second important aspect of the relationship is that what it is *to be* a man is to be an animal, but not vice versa. And so, even though *star* covers a larger field than *man* (there are more stars than men), *man* is not included in *star* (this is so because what it is to be a man has nothing to do with being a star). Thus, the second criterion just mentioned—namely, A is included in B just in case what it is *to be* an A is to be a B, but not vice versa—looks to be essential to the concept of class inclusion. It is worth noting that Aristotle says, 'species and genus do not merely indicate quality, like the term "white"; "white" indicates quality and nothing further, but species and genus determine the quality with reference to a substance'.[41] So, because *man* and *animal* determine quality with reference to a primary substance (for example, Socrates), they are authentic examples of things *said of* a subject (they are examples of a genuine species and genus). Given Aristotle's example of *white* here, it appears that he thought that white does not determine quality with reference to a primary substance. In other words, *white* does not determine the nature or essence of an individual thing. And so, strictly speaking it is not a species, at least it is not a species proper.

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[34] Aristotle, *Categoriae*, Chapter 5, 3b 15–17.
[35] Ibid., Chapter 5, 2a 11–20.
[36] *Categories*.
[37] Ibid.
[38] Ibid., Chapter 4, 1b 10–16.
[39] Ibid., Chapter 5, 3b 20–21.
[40] Ibid., Chapter 5, 3b 23–4.
[41] Ibid., Chapter 5, 3b 17–20.

Consider another remark, this time in *Posterior Analytics*, which I think is connected to implication:

> I call 'true in every instance' what is truly predicable of all instances—not of one to the exclusion of others—and at all times, not at this or that time only; e.g. if animal is truly predicable of every instance of man, then if it be true to say 'this is a man', 'this is an animal' is also true.[42]

Man's being a member of the class *animal* is here cast in terms of 'animal' being predicable of every member of the class *man*. I shall return to this in time. What I want to stress here is the basic idea of individuals and their being members of classes, and the idea that implication is captured by a scheme of classes included in classes.

In contrast to *said of*, something B is 'in' another A if, and only if, B cannot exist apart from the subject A.[43] *Man* and *animal*, though *said of* Socrates are not *in* Socrates. This is so because *man* and *animal* can exist independently of Socrates—for, they are also said of Plato, who exists independently of Socrates. To be sure, there must be *some* primary substance to which *man* and *animal* are predicated, but it need not be Socrates. What is *in* Socrates is whatever depends on Socrates for its existence. Aristotle offers the example 'Socrates is white.' The particular instance of this quality *white* depends for its existence on Socrates. Destroy Socrates and we destroy this particular instance of white. To be sure, Aristotle recognizes that 'white' can be taken to denote a class, as when it is said that 'colour is the genus of white'.[44] Even so, he appears to reject this way of talking. A man who 'says that white is "coloured" has not rendered "coloured" as its genus, seeing that he has used an inflected form, nor yet as its property or its definition'.[45] Although he does not mention it in the passage just cited (*Topics*, Book II), it seems that the criterion given in the *Categories* still applies: since *white* is something that does not determine quality with reference to a primary substance (which, as suggested earlier, means something like—it does not determine the essence or nature of a thing), it does not technically count as a species proper. *White* is best understood as an *accident* of (or *in*) things,[46] and is an item belonging to category (2)—that is, the category: things not said of but in a subject.[47]

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[42] Aristotle, *Analytica Posteriora* in *The Works of Aristotle*, vol. I, edited by W. D. Ross, translated by E. M. Edghill (Oxford: Oxford University Press, 1928), 73a 27–31.

[43] *Categories*, 1b 20–24.

[44] Aristotle, *Topica* in *The Works of Aristotle*, vol. I, edited by W. D. Ross, translated by E. M. Edghill (Oxford: Oxford University Press, 1928), 109a 35–7. He also considers *white* in terms of a species in Book III, 123b 25–30. One can see, then, how Aristotle differs from someone like Frege. Where Aristotle will not count white a species proper, Frege, in his theory of concepts, will. For Frege's view see his discussion in 'On Concept and Object', translated by P. T. Geach and revised by Max Black, in *Mind* (New Series), vol. 60, no. 238 (Apr. 1951), pp. 168–80.

[45] *Categories*, 109b 7–10.

[46] Ibid., 102b 7–13.

[47] Along Fregean lines, I see no reason to say that *white* has no extension, for '*x* is white' certainly has an extensional class (which will even have members).

As it was known to Aristotle, analysis 'reflected principally its function of systematizing geometrical knowledge and coordinating results by leading propositions back to first principles'.[48] But, in both the *Anterior* and *Posterior Analytics*, and in the *Topics*, he discusses analysis with an eye towards developing a method for the systematization of knowledge more generally—that is, he does not concern himself with geometry per se, though he does use examples from geometry to make out the general idea of systematization. In these discussions it is the method of the sorting of classes—determining which are related, which are included in which, and so on—that counts as systematizing. For, in arriving at the proper sorting of classes, where the classes are taken to determine quality with respect to primary substance, one arrives at the proper sorting of the primary substances. But, what is more, the resulting system of inter-related classes expresses what I will call the *conceptual landscape*. In line with Plato, then, analysis for Aristotle is really a method of *sorting*—it is a method of collection and division. Given that there are at least two mutually exclusive classes, and all of the objects being investigated are *collected* into either one or the other (but not both), the objects will be properly *divided* up.

Sayre works out a sophisticated logical machinery that appears to be implicit in Plato's development of analysis.[49] In the *Sophist*, for example, Sayre finds an important feature of the method, namely what he calls *conceptual entailment*.[50] He says:

> $$X \rightarrow ABC. \tag{1}$$
>
> The reading of 'X → ABC' is 'X entails A and B and C', the meaning of which is that any thing conceived as being X is thereby conceived as being A, B, and C as well.
>
> If the combination of features ABC is sufficient for being X, then there is also the relation:
>
> $$ABC \rightarrow X. \tag{2}$$
>
> The reading of (2) is 'A, B, and C together entail X', meaning that anything conceived to be A, B, and C is conceived thereby to be X as well. If the combination of features ABC is both necessary and sufficient for being X, then the following relation holds:
>
> $$X \leftrightarrow ABC. \tag{3}$$
>
> This latter expression can also be written:
>
> $$X = ABC, \tag{4}$$

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[48] Gulley (1962), p. 15.

[49] Sayre (1969), Chapter 4.

[50] Ibid., p. 217f.

which is an obviously appropriate way of symbolizing the claim that X is to be defined as ABC.[51]

Although I think that Sayre is, in the long run, closer to capturing Plato's method of analysis than is Robinson, to my lights he does not fully capture it. But, let's start with what I think is right. If we discover that being ABC is both necessary and sufficient for being X, then we have in our possession a definition of X, namely 'ABC'. A, B, and C, are supposed to be features. So, where 'X' = 'man', and 'A' = 'rational', 'B' = 'two-footed', and 'C' = 'animal', we arrive at the definition that 'man = a rational, two-footed animal'.[52] So far, so good.

Trouble emerges, I think, in Sayre's emphasizing that 'There is no limit theoretically to the number of combinations of features which might be sufficient for being a certain thing or kind.'[53] For Sayre, as this last quote reveals, the analysis of *man* shows that man is simply the *combination* of features A, B, and C. But, there is an important relation holding between A and C, for example, that is not shown in the formula 'ABC', namely, the relation of class inclusion. *Rational* is a class, and in the case of man, it is a class that is included in the class *animal*. The classes are not *combined* by way of conjunction, but exist, so to speak, on entirely different ontological (or even conceptual) levels—with respect to determining the nature of a primary substance (in this case an individual man) *rational* is a species whereas *animal* is its genus. To be sure, Aristotle will not agree with Plato about the ontological status of such things as ideas (that is, *species* and *genus*), but they would seem to be in complete agreement that, in the case of the analysis of *man*, these ideas are *conceptually* related by something more than mere conjunction. In the case of *man*, when properly understood, *animal* and *rational* form part of a hierarchy of ideas—*animal* being 'higher' in the hierarchy than *rational* insofar as the former contains or includes the latter, in the sense of 'includes' discussed earlier. And so, the proper *logical* relation between these features is not conjunction, but is expressed by the *conditional*.

According to Sayre, analysis for Plato 'involves the distinct procedures of discerning common elements in diverse forms of the thing to be defined and of articulating a formula specific enough to mark this off from all other things'.[54] This, I think, is right. Even so, Sayre does not work out the philosophical underpinnings of the procedure. Although I will say much more about this in Part II, it will be helpful to say here that Plato's procedure for collecting and dividing is underwritten by the principle *like attracts like*,[55] or as I shall call it, *like with like*, which is traceable to the Presocratics. Simplicius tells us that:

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[51] Ibid., pp. 217–18.

[52] Sayre (1969) considers this very case on p. 222.

[53] Ibid., p. 221.

[54] Ibid., p. 224.

[55] He raises it, for instance, in *Republic*, Book IV.

> Theophrastus says that the theory of Anaxagoras resembles that of Anaximander; for Anaxagoras says that, in the dividing up of the infinite, things of a like kind tend together.[56]

The idea, I think, is that things are sorted into groups in virtue of their being similar to one another. Take for example a bunch of marbles (let's say that there are a hundred) scattered on the floor. We could sort them by applying '*x* is the same colour as *y*' (this is an instance of the *like with like* principle). Thus, if P and Q are marbles, and 'P is the same colour as Q' is true, they are placed in the same jar. Assuming for sake of simplicity that each marble is monochromatic, after following the procedure just introduced, we would end up with several jars—a jar for each colour of marble. And so, assuming that the colours are primary colours here, ideally we would have three jars, one containing only red marbles, one containing only blue marbles, and one containing only yellow marbles. The idea behind *like with like* seems to be that it works to separate objects into classes—red, blue, yellow in this case.[57] *Collection* is satisfied by our filling the jars, *division* by our establishing three distinct, mutually exclusive jars.

Note that there are several things established by way of the above procedure. First, as was just noted, we establish three mutually exclusive classes of object. Second, given the simplicity of the case (primary coloured marbles on the floor), every marble on the floor will be in exactly one jar, which is to say, each object of investigation is a member of exactly one of the mutually exclusive classes. What is more, in applying '*x* is the same colour as *y*', we also establish what I will call a *vertical* distinction. Here is what I have in mind. In performing the above procedure we produce the following scheme:

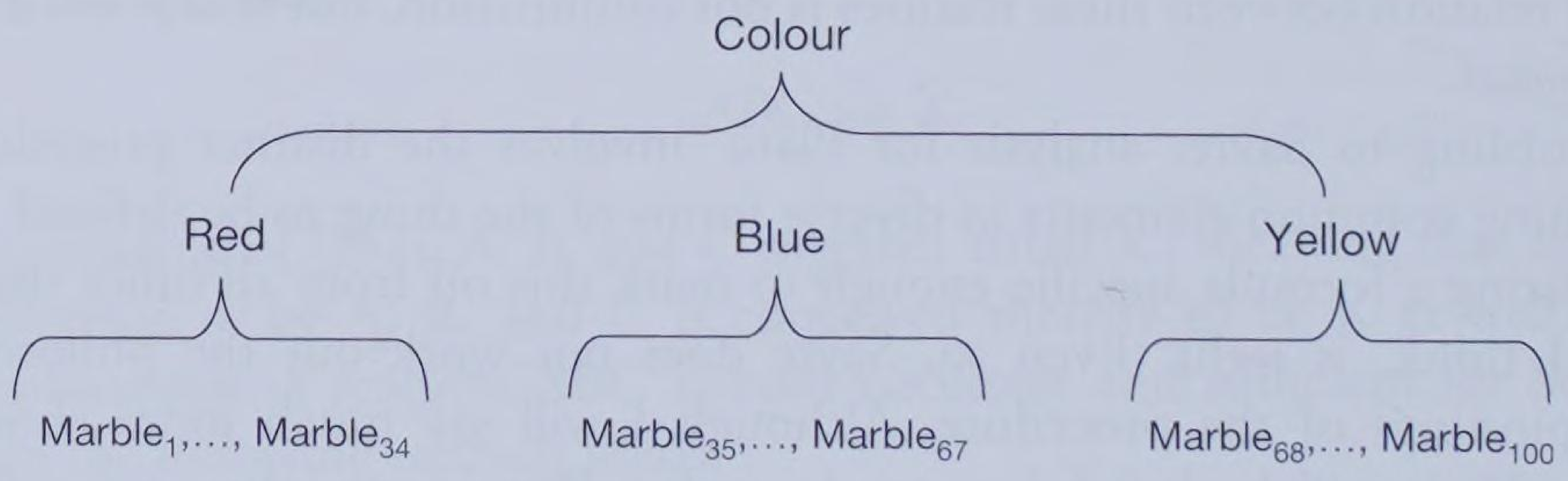
Colour
Red
Blue
Yellow
Marble$_1$,..., Marble$_{34}$
Marble$_{35}$,..., Marble$_{67}$
Marble$_{68}$,..., Marble$_{100}$

Figure 2.1

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[56] Anaxagoras, fragment 492, in *The Presocratic Philosophers*, 2nd edn., G. S. Kirk, J. E. Raven, and M. Scholfield (Cambridge: Cambridge University Press, 1957), p. 373.

[57] For Anaxagoras, the separation would be into hot, cold, moist, and so on.

Note that in Aristotelian terms,[58] marble$_1$, marble$_2$, and so on, being the individuals here, count as primary substances; Red, Blue, and Yellow as species or secondary substances; Colour as the genus (or tertiary substance). Marble$_1$, marble$_2$, marble$_{66}$, and so on, are *horizontally* distinct from one another, since we find them at the same level of the structure. However, Red, for instance, is *vertically* distinct from marble$_1$ and marble$_2$: Red's containing the latter puts Red 'higher' in the structure. Now, Red, Blue, and Yellow are *horizontally* distinct from one another—they are items at the same structural level. Colour, on the other hand, is 'higher' in the structure. The distinction between Colour and Red, Blue, and Yellow is vertical—thus, Colour and Red (or Colour and Blue or Colour and Yellow) are *vertically* distinct. As we just did with marble$_1$ and Red, and as with the earlier case of *animal* and *man*, we can express the vertical distinction as follows: the class Colour includes or contains the classes Red, Blue, and Yellow. Thus, our procedure of collection and division established in one fell swoop a logical (or conceptual) structure, which is a region of the conceptual landscape. This sort of procedure, I contend, is what Plato and Aristotle have in mind when performing analysis.

The structure that we establish expresses (or shows) certain logical relations that underwrite what above we were calling *implication*.[59] For example, according to the structure, 'All red things are coloured things' is true, but not vice versa. (And so, Robinson's insistence on *reciprocal* implication is again called into question.) The thing to stress here is that the structure expresses: 'If something *x* is red, then that something *x* is coloured.' Consequently, our Aristotelian view worked out earlier tells us that if marble$_1$, say, is a member of the class denoted by 'Red', then 'marble$_1$ is a red thing' is true. And, if the class *Red* is included in the class *Colour*, then whatever is a member of *Red* is a member of *Colour*. Thus, again according to our Aristotelian view, 'Colour' is predicable of marble$_1$. And so, 'marble$_1$ is a coloured thing' is true. The syllogism expressing the structure is:

All red things are coloured things.
Marble$_1$ is a red thing.
So, marble$_1$ is a coloured thing.

Implication is a concept that is brought into play as a result of doing analysis, but it is not the *point* of doing analysis. Rather, the point of doing analysis seems to be the working out of a conceptual structure (which in turn constitutes some region of the larger conceptual landscape).

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[58] Since Aristotle might reject the idea that *red* and the other colours determine quality with respect to a primary substance, the example is not perfect. Assume for the sake of making the idea clear here that *red*, *blue*, and *yellow* do determine quality with respect to a primary substance.

[59] See Taylor for more on the connection between inference and class-inclusion. A. E. Taylor, *Aristotle* (New York: Dover Publications, 1955), pp. 28ff.

Aristotle showed a great deal of interest in such structures, especially in the idea of locating predicables at their proper levels in the hierarchy. For instance, in *Posterior Analytics* he says:

> An attribute belongs commensurately and universally to a subject when it can be shown to belong to any random instance of that subject and when the subject is the first thing to which it can be shown to belong. Thus, e.g. . . the equality of its angles to two right angles is not a commensurately universal attribute of figure. For though it is possible to show that a figure has its angles equal to two right angles, this attribute cannot be demonstrated of any figure selected at haphazard, nor in demonstrating does one take a figure at random—a square is a figure but its angles are not equal to two right angles. On the other hand, any isosceles triangle has its angles equal to two right angles, yet isosceles triangle is not the primary subject of this attribute but triangle is prior . . . [where] it is of wider application.[60]

The above structure seems to be (italicized names denote classes, where we can denote any member of a class by the class's name—so, a member of the class *figure* can be called a 'figure', and so on): *Figure*, which includes such classes as *triangle* and *square*; and *triangle*, for instance, which includes such classes as *isosceles*. Locating 'equality of its angles to two right angles' is not predicable of figure *as figure*, since it is not true of every (member of) *figure*.[61] If we randomly choose *square*, which is a member of *figure*, we discover that the above is not predicable of any member of *square*. On the other hand, if we choose *isosceles*, we will discover that it *does* hold for every member taken randomly. So, does 'equality of its angles to two right angles' belong commensurately and universally to isosceles?

It may seem so at first, but notice that there is something more than simply belonging to any member taken randomly—it must belong to the *first thing* to which it can be shown to belong. Is *isosceles* the first thing? Suppose that we now choose *scalene*, which is another member of *triangle*. The property in question will be discovered to belong to any member of it too. What about *equilateral*? It is also a member of *triangle*, and the property belongs to any member of it too. What Aristotle seems to have in mind is that the property belongs to these three species of triangle, but it does so in virtue of their being triangles. When compared to classes of its own level—that is, when compared to *circle*, *square*, and so on—*triangle* is the only member of figure whose own members possess the property. *Triangle* is the first thing in the hierarchy to possess the property. Thus, 'equality of its angles to two right angles' belongs commensurately and universally to figure *as triangle* (and neither *as figure*, which turns out to be too broad, nor *as isosceles*, which turns out to be too narrow).

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[60] *Analytica Posteriora*, 73b 32–74a 45.

[61] I intend the discussion here to be loose enough so that we could easily move between our saying 'P is not predicated of *Figure*', and 'P is not predicted of any member of *Figure*', and 'P is not predicated of a figure', and so on. In his commentary on Aristotle, Aquinas makes it clear that Aristotle draws the distinction between the *class* man, for example, and *a* man. He says, 'This flesh and these bones and this complexion are not implicit in the definition of man, nevertheless they are included in a thing called man. Therefore that thing which is a man has in himself something that humanity does not have, and for this reason a man and humanity are not convertible terms.' *St. Thomas Aquinas: Philosophical Texts*, translated by Thomas Gilby (New York: Oxford University Press, 1960), pp. 158–9. This passage comes from *Summa Theologica*, Ia, iii, 3. So, thinking about the geometrical case under discussion, Aristotle should not be taken as being vulnerable to the error of attributing three-angledness to the *class* of triangle.

The test for locating a predicable in the hierarchy, then, is twofold. First, where P is a predicable:

> P belongs commensurately and universally to members of class C if, and only if,
> (a) any member of C randomly chosen will be shown to possess P,
> and,
> (b) P will not be possessed by any members of any other classes found at the same level of *C* in the hierarchy.

And so, here is the conceptual structure (represented by this scheme):

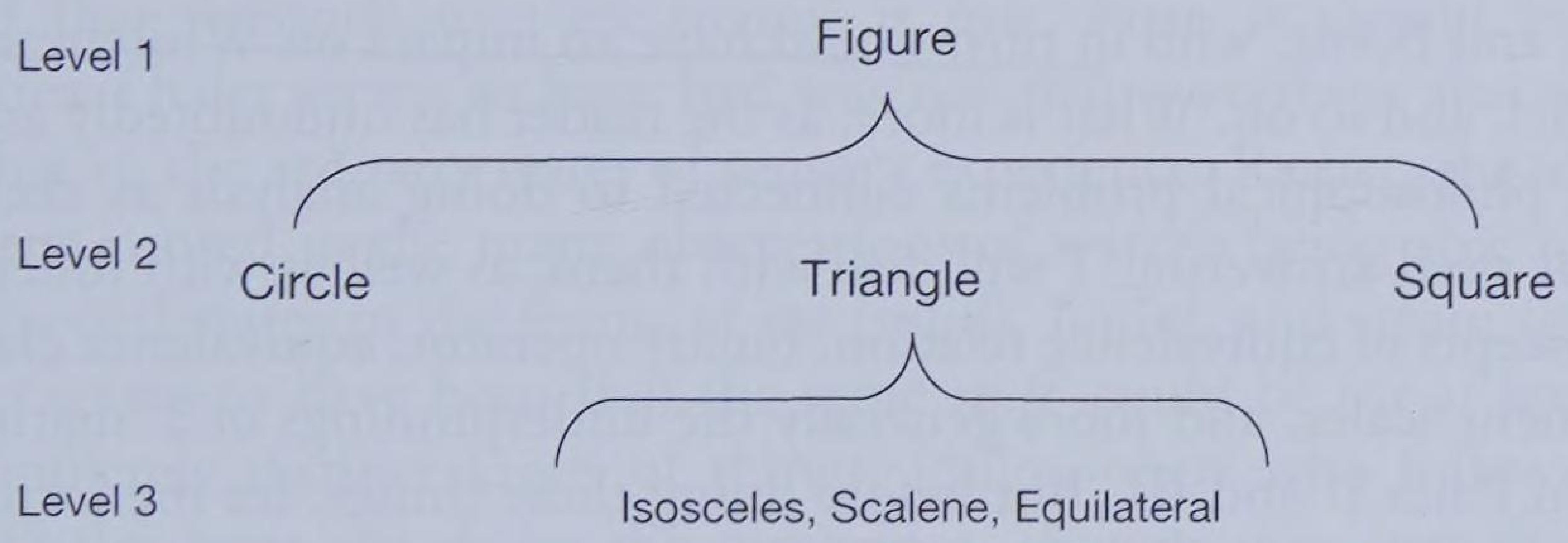

Figure 2.2

We could extend level 2 to include pentagons, hexagons, and so on. I include only the three kinds of shape here (in Figure 2.2) for the sake of discussion. When we take something *as figure* and ask whether 'equality of its angles to two right angles' belongs commensurately and universally to this thing *as figure*, we discover that if we were to choose any member of *figure* other than *triangle* (that is, at level 2), the property would fail to belong. And so, in failing to meet criterion (a), we say that the property does not belong commensurately and universally to a thing simply as *figure*. When we take this thing *as isosceles* and ask the same question, we discover that if we were to randomly choose any member from *isosceles*, the property belongs to it. The trouble is that if we were to choose any member from *scalene* or *equilateral* (that is, from other members of level 3), the property will be found to belong to them too. The property, in other words, is not unique to members of *isosceles*. And so, in failing to meet criterion (b), we say that the property does not belong commensurately and universally to a thing as *isosceles* (or, to isosceles triangles). The only thing that meets both criteria is *triangle*. Thus, the property belongs commensurately and universally to the thing as *triangle*, and is a predicable located at level 2 of the hierarchy.

Developing and clarifying the idea of the above sort of structure, and its connection to implication, would make a great impression on philosophers of the early modern period. And, from their hard work, the connection would greatly impress philosophers of the nineteenth and twentieth centuries. For example, echoing what was said above, Bertrand Russell says in *Principles of Mathematics*, that 'pure mathematics is the class of all propositions of the form "*p* implies *q*" ', holding implication to be 'the relation of a term to a class of which it is a member'.[62] Robert Carmichael claims that 'a set of objects, with the associated rule or rules of combination, is called a *system*, or more explicitly, a *mathematical system*.'[63] And, even though Plato and Aristotle might not have seen the connection between what they were producing via analysis—namely, a structure of classes by way of applying the combinatorial rule *like with like*—and mathematics, the fact is that the connection was there. As I hope to show in the chapters to follow, this connection would be something that in part was being discovered by Descartes, Leibniz, Pascal, and others of the early modern period, who in turn would have an impact on Euler, Gauss, Abel, Galois, Cayley, Poincaré, and Boole, who in turn would have an impact on Whitehead, Russell, Carmichael, and so on. What is more, as the reader has undoubtedly ascertained, there are philosophical problems connected to doing analysis as sketched out above that need answering. I will deal with them, as well as with fuller accounts of the concepts of equivalence relation, binary operator, equivalence class, group, measurement scales, and more generally the underpinnings of a 'mathematized' physics, in Parts II and III. But before doing these things, let me introduce one last topic crucial to my overall project.

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[62] Bertrand Russell, *The Principles of Mathematics* (Cambridge: Cambridge University Press, 1903), p. 3.

[63] Robert Carmichael, *Introduction to the Theory of Groups of Finite Order* (New York: Dover Publications, 1956—originally published by Ginn and Company, 1937), p. 4.

3

Matter and Division

In the *Metaphysics* Aristotle gives a brief history of the concept of matter. Here is a part of that history:

> Most of the first philosophers thought that principles in the form of matter were the only principles of all things; for the original source of all existing things, what from which a thing first comes-into-being and into which it is finally destroyed, the substance persisting but changing in its qualities, this they declare is the element and first principle of existing things, and for this reason they consider that there is no absolute coming-to-be or passing away, on the ground that such a nature is always preserved . . . Thales, the founder of this type of philosophy, says that it is water.[1]

The word that Aristotle uses for matter is *hyle*. First, it should be stressed that the idea Thales seems to have had was not that everything was ultimately water—that is, the *ordinary* water of sensory experience. Rather, the idea seems to have been rooted in the many observations of water's being able to take on distinct physical states in the forms of ice (solid), liquid, and steam (gas).[2] The idea, then, seems to have been that the same *stuff* could be (or at least appear to be) completely distinct kinds of things. Philosophers who followed in the wake of Thales were struck by the same idea, though they cast this stuff in different terms—Anaximenes casting it as *air*,[3] Anaximander as *the indefinite* (*to apeiron*),[4] and so on. In the *Physics*, Aristotle himself seriously entertains the idea that matter (*qua* primary underlying stuff) is the genuine subject of change, brought about by taking on various forms (various characteristics),[5] though as several commentators note, the simplest instances of 'natural bodies' for Aristotle appear to be the elements—earth, fire, air, and water.[6]

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[1] The translation of this passage is 'Thales-Fragment 85', in Kirk, Raven, and Scholfield, p. 89. The original passage comes from *Metaphysics* A3, 983b 6.

[2] In Fragment 87, Heraclitus Homericus refers to water's rarefaction and becoming air (gas), and to its compaction and becoming earth (solid); Kirk, Raven, and Scholfield, p. 92 (fn. 2, continued from p. 91).

[3] Fragment 139, found in Aristotle's *Metaphysics*, A3, 984a 5, in Kirk, Raven, and Schofield, p. 145.

[4] See Kirk, Raven, and Schofield discussion, pp. 105ff.

[5] Aristotle. *Physics*, 192a 31–2.

[6] For example, see Sir David Ross, *Aristotle* (London: Methuen & Co Ltd, 1923), p. 66, and A. E. Taylor, *Aristotle* (New York: Dover Publications, 1955), p. 71.

In the *Timaeus*, Plato provides an account of the origin of the cosmos, which has Anaxagorean and Pythagorean undertones, an account with which Aristotle was undoubtedly familiar.[7] Although Plato does not refer to it as such, the *receptacle*, which acts as medium for the becoming of all observable things, plays what appears to be the role of matter. There are three items in play: (1) the observable particular (that which becomes); (2) the receptacle (that in which it becomes); and (3) the form (the model in whose likeness that which becomes is born).[8] Plato relates the three as follows:

> It [the receptacle] must be called always the same; for it never departs at all from its own character; since it is always receiving all things, and never in any way whatsoever takes on any character that is like any of the things that enter it: by nature it is there a matrix for everything, changed and diversified by the things that enter it, and on their account it *appears* to have different qualities at different times; while the things [the observables] that pass in and out are to be called copies of the eternal things [the forms], impressions taken from them in a strange manner that is hard to express.[9]

About the receptacle, Donald Zeyl says:

> In its own right it is (part of) a totally characterless subject that temporarily in its various parts gets characterized in various ways. This is the receptacle—an enduring substratum, neutral in itself but temporarily taking on the various characterizations. The observed particulars just are parts of that receptacle so characterized.[10]

In light of Zeyl's interpretation, I read the above *Timaeus* passage as saying that an observable is an informed region of the receptacle. Thus, the world of individuated bodies is ultimately the receptacle 'divided up' by way of a complex system of forms—the forms dividing the plenum into kinds of things, the matter serving as the *medium of instantiation*, which, in an unexpected foreshadowing of Aristotle, seems to serve as an individuating principle. For, there is now a sense in which the kind *man*, for example, can be instantiated *here and now* and *there and now*, where 'here' and 'there' refer to distinct spatial locations (made possible by the receptacle). Thus, there is a sense in which we can talk of *two* men, and not simply of *man*.

A bit later, related to the three items, Plato says:

> We must agree that there is, first, the unchanging Form, ungenerated and indestructible, which neither receives anything else into itself from elsewhere . . . [it is] that, in fact, which thinking has for its object.

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[7] According to Cornford, 'Aristotle was living and working with Plato when the dialogue [*Timaeus*] was written.' Francis M. Cornford. *Plato's Timaeus* (New York: The Liberal Arts Press, 1959), translator's preface, p. xii.

[8] *Timaeus*, 50c–d, 52a–b. I am using Cornford's translation and commentary in his *Plato's Timaeus* (New York: The Liberal Arts Press, 1959). For more on the controversy over taking the receptacle as matter, see Zeyl (2005).

[9] *Timaeus*, 50c.

[10] Donald Zeyl, 'Plato's Timaeus', *The Stanford Encyclopedia of Philosophy* (October 2005), section 6; http://plato.stanford.edu/entries/plato-timaeus.

> Second is that which bears the same name and is like that Form; is sensible; is brought into existence; is perpetually in motion, coming to be in a certain place and again vanishing out of it . . . [it is an object] of perception.
>
> Third is Space,[11] which is everlasting, not admitting destruction; providing a situation for all things that come into being, but itself apprehended without the senses.[12]

The order of introduction is the form, the observable, and the receptacle. Our earlier discussion of the Visible and Intelligible sections of the divided line should come to mind here. No doubt, the observable is an example of a Visible, and the form an example of an Intelligible. Given that the receptacle is not apprehended by the senses, the alternative seems to be some kind of intellection. Plato, in fact, calls it a 'sort of bastard reasoning',[13] which suggests, I think, that it is not understood in the same manner in which the forms are understood. I want to suggest that in light of the divided line analogy—where, recall, forms were objects understood under the mental state of *noesis*, and observables at best under the mental state of *pistis*—the receptacle is that which is understood under the mental state of *dianoia*. It comes as part of the production of mathematical objects, which, recall, is done via our regarding the observables in a certain way.

At least one Muslim commentator of Aristotle drew a picture that shows some interesting overlap to the one we find in Plato's *Timaeus*—similar enough, I think, to at least justify our entertaining the idea that Plato and Aristotle agreed, at least in part, on the role of matter, its 'division' into kinds by way of the forms, and how this resulted in the production of particular observables—matter serving as the metaphysical medium through which the forms acted. For example, in *The Letter Concerning the Intellect*, Alfarabi says that the forms, prior to their being *thought* by an intellect, are 'intelligibles in potentiality'.[14] When they are intelligibles in potentiality they 'exist in themselves [as forms of matter]', where enmattered they exist 'outside the soul'.[15] They are 'intelligibles in *actuality*' only upon their being abstracted from a particular observable.[16] And, once abstracted they are coextensive with thought itself.[17]

Although it is unclear, Alfarabi seems to have thought that what he calls the *agent intellect*, which exists independently of matter, is responsible for imparting the forms in matter originally.[18] It is also responsible for activating the power (*potentia*) of the intellect to abstract those forms from matter.[19]

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[11] Here, Cornford renders *choras aei* as 'space', but R. G. Bury better renders it 'ever-existing place', in his translation of *Timaeus*, in *Plato*, vol. vii, (Cambridge: Harvard University Press, 1929), p. 123.

[12] Ibid., 51d–52c.

[13] Cornford renders *logismo tini notho* as 'bastard reasoning'. *Logismos* can also be rendered 'reckoning' or 'computation'—in other words, we might read this as connected to Plato's earlier reference to 'calculating reason'. See earlier footnote 1, Chapter 1.

[14] Alfarabi, *The Letter Concerning the Intellect*, in *Philosophy in the Middle Ages*, translated by Arthur Hyman (Indianapolis: Hackett Publishing, 1973), p. 216.

[15] Alfarabi (*The Letter*).

[16] Ibid., pp. 215–16.

[17] Ibid., p. 217.

[18] Ibid., pp. 218–19.

[19] Alfarabi (*The Letter*).

Now, an interesting question here is, what ontological status does Alfarabi give the forms *prior* to their being imparted to matter? Given his view that the intelligible in actuality is coextensive with thought itself, arguably prior to its being imparted to matter, the intelligible was (ontologically) located in the agent intellect; suggesting, I think, that the system of forms simply *is* the conceptual landscape—the cognitive structure—of the agent intellect, something that both Augustine and Aquinas adopted from their exposure to Neoplatonism.[20]

Although we can find the above sorts of overlap, according to A. C. Crombie there are differences between Plato (or, at least the Neoplatonists) and Aristotle that should be stressed:

> The main difference between Aristotle's view of matter and that which had been put forward by such Neoplatonists as St. Augustine and Erigena concerned the nature of the substance that persisted through substantial change. For these Neoplatonists this persisting substance was actual extension, that is pure potentiality (or *materia prima*) determined by spatial dimensions, and this underlay all other attributes of material things; for Aristotle it was simply pure potentiality. With some Arab philosophers such as Avicenna, Al-Ghazzali and Averroës and the Spanish Rabbi Avicebron, the Neoplatonic theory of matter took the form that every material thing possessed a 'common corporeity' making it extended, and, according to Avicebron, this corporeity was continuous through the universe. The importance of this theory was that it introduced the possibility of extending the use of mathematics to the whole of natural science.[21]

A good example of such a use of mathematics, Crombie notes, is found in Robert Grosseteste's mathematical treatment of light, having identified the Neoplatonic concept of common corporeity with light.[22] Echoing the *Timaeus*' 'expansion' and subsequent work of the World-Soul, Grosseteste claims that 'the universe arose from a point of light which by auto-diffusion generated the spheres of the four elements and the heavenly bodies and conferred on matter its form and dimensions.'[23]

Crombie mentions John Scottus Eriugena, who lived during the ninth century, coming close to bisecting the segment of timeline holding between Augustine and Aquinas. The only work of Plato's translated into Latin available at the time of Eriugena was in fact the *Timaeus*. Its impact on Eriugena becomes clear in the title of what many consider to be his greatest work: *On the Division of Nature*.[24] Guided in part by Augustine's view of creation,[25] Eriugena sees the cosmos as originally being divided up by God. According to Henry Bett, the method Eriugena employs to make the division is analysis (*analutike*), the resulting conceptual structure showing how the reality descends 'from the most general to the most special . . . resolving individuals into species, species into genera, genera into essences, and essences into the wisdom of the Deity . . . whence all these divisions arose and where they end'.[26] In Part III, we will see one consequence of Eriugena's work in the ideas of Raymond Lull, a twelfth-century Spanish theologian, whose work in combinatorics would in turn have a palpable impact on Descartes and Leibniz.

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[20] For an example of Aquinas' adoption of Augustine's divine exemplar view, see *Summa Theologica*, Ia, lxxxiv. 5; for divine emanation, see xiv. 1 and Ia, xliv. 2 *ad* 1.

[21] A. C. Crombie, *Augustine to Galileo*, 2 vols (Cambridge: Harvard University Press, 1961), vol. 1, p. 73.

[22] Crombie (1961), p. 74.

[23] Crombie (1961).

[24] For such assessments, see for example, Henry Bett, *Johannes Scotus Erigena* (New York: Russell & Russell, 1964), p. 19ff.

[25] Bett (1964), p. 21.

Let's return to Plato. We find in the *Timaeus* his bringing to light certain necessary conditions underwriting the creation of the cosmos relevant for the story that I want to tell:

> Now that which comes to be must be bodily, and so visible and tangible; and nothing can be visible without fire, or tangible without something solid, and nothing is solid without earth. Hence the god, when he began to put together the body of the universe, set about making it of fire and earth. But two things alone cannot be satisfactorily united without a third; for there must be some bond between them drawing them together. And of all bonds the best is that which makes itself and the terms it connects a unity in the fullest sense; and it is the nature of a continued geometrical proportion to effect this most perfectly. For whenever, of three numbers, the middle one between any two that are either solids (cube?) or squares is such that, as the first is to it, so is it to the last, and conversely as the last is to the middle, so is the middle to the first, then since the middle becomes first and last, and again the last and first become middle, in that way all will necessarily come to play the same part toward one another, and by so doing they will all make a unity.[27]

According to Plato, two elements—fire and earth—are the first elements of creation. They must be relatable. Otherwise nothing else can arise. The relation that Plato sees as the best is *proportion*. He goes further, speaking in terms of numbers and proportion, and tells us why there are not simply two but four elements:

> Now if it had been required that the body of the universe should be a plane surface with no depth, a single mean would have been enough to connect its companions and itself; but in fact the world was to be solid in form, and solids are always conjoined, not by one mean, but by two. Accordingly the god set water and air between fire and earth, and made them, so far as was possible, proportional to one another, so that as fire is to air, so is air to water, and as air is to water, so is water to earth, and thus he bound together the frame of a world visible and tangible.[28]

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[26] Bett (1964), pp. 21–2.

[27] Plato, *Timaeus*, translated by Benjamin Jowett, in *Plato: The Collected Dialogues*, edited by Edith Hamilton and Huntington Cairns (Princeton: Princeton University Press, 1989), pp. 1151–211. The relevant citation is 31b—31c.

[28] *Timaeus*, 31c—32c.

The world has depth; it is solid. It is not just a plane

So, though elements in their own right, air and water serve as means holding between fire and earth. As Plato says, were the body of the cosmos (Jowett translates *to tou pantas soma* as 'frame'[29] and Bury as 'the body of the All'[30]) only a surface, the two elements would require only one mean between them. But, since the frame is a solid—that is, it is three-dimensional—there must be two means.

In the first of the two passages just cited, Plato mentions the solid (or cube) and the square (a plane figure). The latter requires only one mean; the former requires two. There are several ways to picture what Plato may have in mind. One, I think, should be familiar to today's philosopher,[31] and is something that we will revisit in Part III. Consider the mathematical formula $(x + 1)^2$. Let x be one element, 1 the other, and let the exponent tell us that two dimensions are in play. If we do the combinatorial work to expand this, we get: $x^2 + 2x + 1$. Note that between the two extreme terms (x^2 and 1) is one (mean) term $2x$. Now, consider the formula $(x + 1)^3$. The exponent here tells us that three dimensions are in play. If we do the combinatorial work to expand this, we get: $x^3 + 3x^2 + 3x + 1$. Note that between the two extreme terms (x^3 and 1) are two mean terms, $3x^2$ and $3x$. If the body (or frame) of the cosmos were a plane, then there would be three elements: fire and earth, and some element that served as a mean between them. But, since the body (or frame) of the cosmos is a solid (it is three-dimensional), Plato tells us that there are four elements: fire and earth, and two elements that serve as means between them. Determining the number of elements appears to have been a matter of combinatorial computation.

The division of the receptacle is accomplished via what Plato calls the *World-Soul*. It starts at the receptacle's centre, dividing the receptacle as it 'extends' or moves outward.[32] Plato does not see the receptacle as being created prior to the World-Soul, however. Introducing them in this order (receptacle and then World-Soul) is simply a 'manner of speaking',[33] brought about by the natural limitations of human explanation. Three concepts are introduced to explain what division amounts to—intermediate existence, intermediate sameness, and intermediate difference.[34] These are 'intermediate' insofar as they hold between two extremes. For instance, intermediate existence holds between 'the indivisible Existence that is ever in the same state and the divisible Existence that becomes in bodies';[35] the same for intermediate sameness and difference, where the god 'made a compound intermediate between that kind of them which is indivisible and the kind that is divisible in bodies'.[36] Although we get no further account as to why there is only one mean between each set of extremes (instead of two, as with the elements), he goes on to say:

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[29] Timaeus.

[30] R. G. Bury, *Timaeus*, p. 59.

[31] I am indebted to Charles Young not only for this particular example, but for bringing to light what would appear to be Plato's use of combinatorics.

[32] *Timaeus*, 34a–34b.

[33] *Timaeus*, Jowett's translation (p. 1165), 34c.

[34] This is Cornford's terminology. See p. 25, fn. 2.

[35] *Timaeus*, 35a.

[36] Timaeus.

> And having made a unity of the three, again he divided this whole into as many parts as was fitting, each part being a blend of Sameness, Difference, and Existence.[37]

Think back to those marbles (introduced earlier). We might understand Plato's point as follows. Recall that we established the structure:

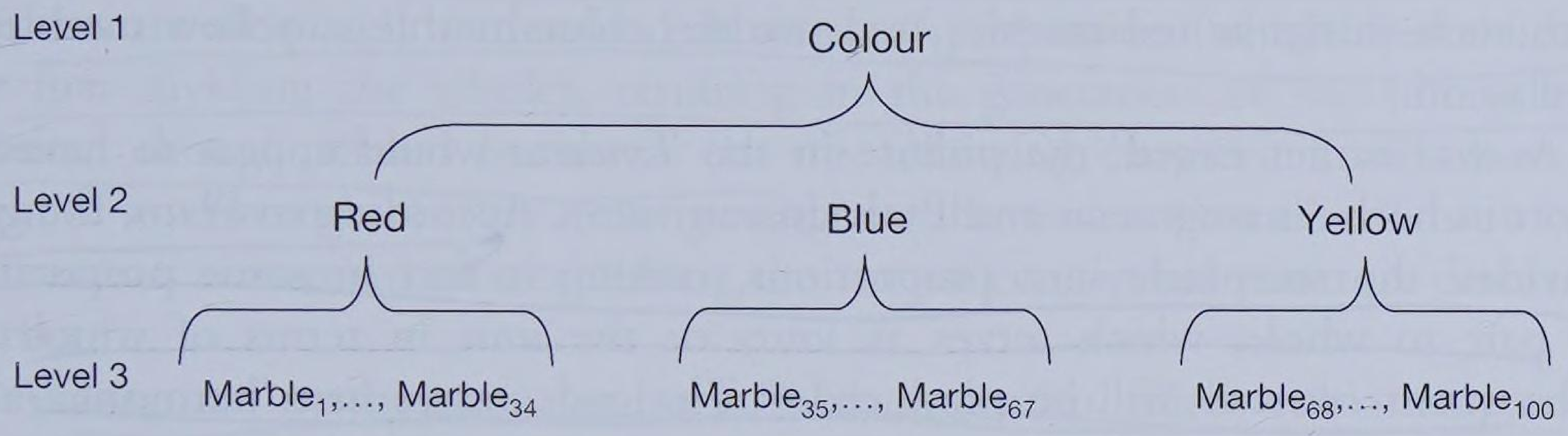
Level 1
Colour
Level 2
Red
Blue
Yellow
Level 3
Marble$_1$,..., Marble$_{34}$
Marble$_{35}$,..., Marble$_{67}$
Marble$_{68}$,..., Marble$_{100}$

Figure 3.1

The marbles that are red cannot be distinguished from one another in terms of being *red*. Each is equally red (think of what Socrates says about those fingers mentioned at the beginning of Chapter 1). That is, if *red* were the only feature we could refer to, then there would be no way to intelligibly make a distinction between marble$_1$ and marble$_2$. At the conceptual level of Red (level 2), where we also find Blue and Yellow, the only differences that *can* be made are these: Red, Blue, and Yellow. So, at this level we can draw an intelligible distinction between marble$_1$ and marble$_{35}$—one is red, one is blue—but not between marble$_1$ and marble$_2$. Even so, since to be red, blue, and yellow is also to be coloured, there is a sense in which there is sameness—for, red, blue, and yellow are colours. Thus, with respect to being coloured, that is, considering things at level 1, there is *no* difference even between marble$_1$ and marble$_{35}$. Both are coloured, and taken solely *as coloured* (as opposed to *red* or *blue*), differences between them are rendered unintelligible. So, it seems that *sameness* and *difference* are relative and are expressed as such by the structure. Going over this once more, then, where Colour at level 1 is taken to express the concept of sameness, Red, Blue, and Yellow, which are at level 2, will be taken to express the concept of difference (this is close to Aristotle's distinction between *genus* and *differentia* or *species*). But, where Red at level 2 is now taken to express the concept of sameness, marble$_1$, marble$_2$, and so on, which are at level 3, will express the concept of difference, though to get this right we would have to find something that allows us to make intelligible the distinction (something that, for sake of discussion, we need not do here).

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[37] *Timaeus*, 35b.

Since Plato takes the forms as the primary substances, a Platonist position will cast the existential component of the cosmos as emanating 'downwards' from level 1 to level 3, where the thing at level 1 is more real than that at level 3. By contrast, an Aristotelian position seems to cast the individual marbles as the primary substances, which 'ground' the entire system, and thus the thing at level 3 is more real than that at level 1. These sorts of differences aside (we will discuss them later), Plato seems to hold that the *intersection* of sameness, difference, and existence ultimately produces the division of the receptacle into such things as red-marble$_1$, red-marble$_2$, blue-marble$_{35}$, yellow-marble$_{83}$, and so on.

As was earlier noted, the picture in the *Timaeus* would appear to have its roots in both Anaxagorean and Pythagorean views. According to Plato, the god 'divides' the receptacle into proportions, cutting it first in some proportion of part to whole, which serves as *unity* or the *unit* in terms of which all subsequent *intervals* will be produced.[38] The god creates first 'harmonic' and then 'arithmetic' intervals by way of inserting them as means between the initial intervals produced (which are unified by their being generated as some proportion of the same unit).[39] According to Shorey, Plato takes the god to divide the two initial parts (those produced as a result of the first division) differently. Shorey sees this as follows:[40]

1 (the 1st)
2 (the 2nd) **3** (the 3rd)
4 (the 4th) **9** (the 5th)
8 (the 6th) **27** (the 7th)

Figure 3.2

The left branch is one part of the receptacle, the right the other. The left branch contains what Shorey calls the 'double intervals', or powers of 2; the right branch contains the 'triple intervals', or powers of 3.[41] Within these, Plato develops and inserts their means. For the double intervals, where 'hm' is *harmonic mean* and 'am' is *arithmetic mean*, he inserts them thus:

1 hm am 2 hm am 4 hm am 8

For the triple intervals, he does the same:

1 hm am 3 hm am 9 hm am 27.

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[38] *Timaeus*, 35b.
[39] *Timaeus*, 35b.
[40] I borrow this from Shorey, pp. 66–7, fn. 2.
[41] Timaeus.

The intervals are thus:[42]

1 4/3 3/2 2 8/3 3 4 16/3 6 8
1 3/2 2 3 9/2 6 9 27/2 18 27

Though fascinating, the details of the remainder of his discussion in the *Timaeus* will only serve to deflect interest from what is important here, so I shall not pursue them further. Enough is clear, I think, for us to make some connections. Let me begin by summarizing Plato's point: the distinctions drawn in the cosmos are ultimately understood in terms of proportion (which is established by first dividing the whole), resulting in the generation of the intervals of distinct though related numbers. (In the following chapters, I will show in light of seventeenth-century dealings with the above view how this division into intervals is connected to establishing the sort of conceptual structure discussed earlier.)

Anaxagoras' cosmogony also begins with a plenum, filled with a mixture of every kind of thing. The challenge to understanding the view is to see that there is no *pure* thing (or kind of thing) for Anaxagoras—though he will talk of seeds (*spermata*), which have been a point of contention among scholars as to whether they counted as pure stuffs.[43] A kind of thing f, he says, is some portion of f to a portion of every kind of thing, where there is a greater portion of f than a portion of any other kind of thing. Let the portion of everything (or every kind of thing) be e (e also includes f). So, kind f is really the proportion or ratio of $f: e$ (or we might put it: $f\ /\ e$), where there is more f than there is a portion of anything else in e. But, if f is really the ratio $f: e$, and there is no pure f, then we can substitute for f its defining ratio $f: e$, getting $(f: e): e$ (or $(f\ /\ e)/\ e$). But, we can again make the substitution, getting $[(f: e): e]: e$ (or $[(f\ /\ e)/\ e]/\ e$), and so on ad infinitum. This looks a bit ugly, but one way to think of this is in terms of numerical ratios. For instance, 1/2 can also be expressed as (2/2)/2, or as [(2/2)/(2/2)]/2, and so on ad infinitum. But for all that, ([(2/2)/(2/2)]/[(2/2)/(2/2)])/2 is still 1/2.

Pythagoras also approached division of reality in terms of proportion, where the example he famously uses to demonstrate the view is the lyre string. If one takes a string and plucks it, one makes a sound (produced by the vibration of the string). Now, divide the string in half (by placing a finger at that spot) and this proportion, 1:2 (or 1/2) produces the octave. Divide the string in thirds (placing the finger on one of the spots marking a third), and the result is the fifth (instead of the octave). So, the ratio 1:2 produces the octave, the ratio 1:3 the fifth. Now, think of that string as a line. The idea, I think, is that there is an infinite number of 'spots', for they are the points on the line. As Aristotle and others understood it—this would be unlike the 'spots' on the string, of course—the points are not sitting side by side (yet they are the elements of the line). The points are not extended, and so they do not have a left and right side, for instance, such that they could be said to 'touch' one another—the right side of point *a* touching the left side of point *b*, the right side of *b* touching the left side of *c*, and so on. But, if the above is right, and the string analogy is taken seriously, how are these 'individual' points related to the line and to one another—especially given that the points are not really like the fingered-spots on a string? One answer suggests that we begin with the whole, that is, the line.

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[42] Again, I should emphasize that I am borrowing completely from Shorey's analysis, p. 68, fn. 1.

[43] For more on the controversy, see Kirk, Raven, and Schofield (pp. 367–8).

So, start with line PQ and make the following arbitrary division PS:

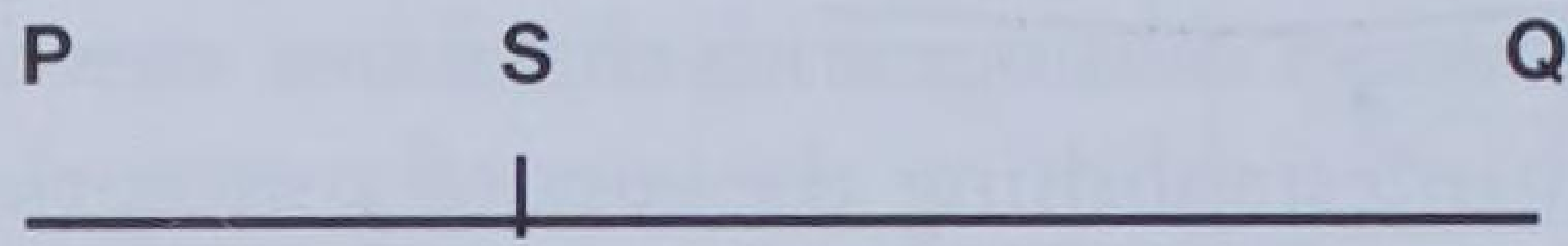

Figure 3.3

The ratio PS:PQ (or PS/PQ) is a unique point on the line PQ. Call this point *a*. Now, divide it again, this time PR, where PS is not identical to PR. So:

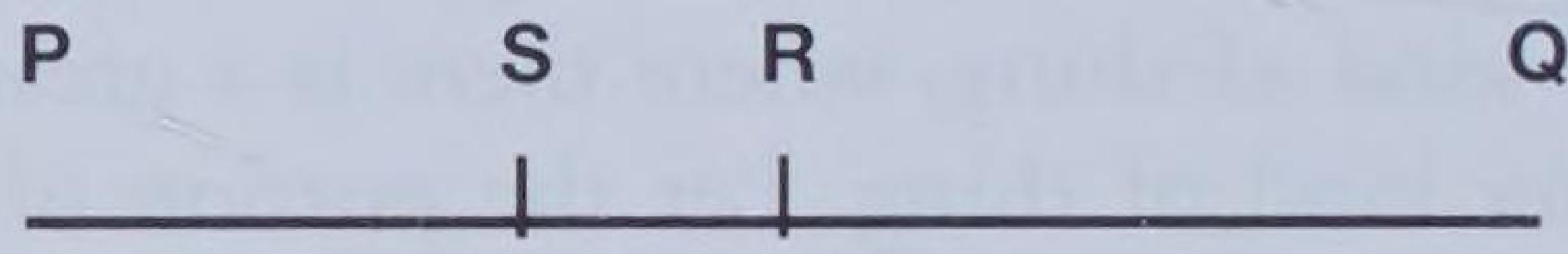

Figure 3.4

The ratio PS:PQ (or PS/PQ) is distinct from the ratio PR:PQ (or PR/PQ), and so the individual point determined by PS:PQ (point *a*) is distinct from the point determined by PR:PQ. Call this latter point, point *b*. So, *a* is distinct from *b*. In Anaxagorean terms, we might think of PQ as 'containing' every point (of which there is an infinite number) and that *a* is some portion PS to a portion of everything PQ; *b* is some portion PR to a portion of everything PQ, and so on. We might say, then, that *a*, *b*, and in fact every point, 'contains' a portion of everything. Something like this, it seems, is also going on in Plato's *Timaeus*, where by way of a division of the receptacle into proportional intervals (the receptacle being analogous to line PQ above), the god produces the *individuals* of the cosmos, each internally related to the others and to the cosmos (the whole).[44]

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[44] By 'internally' related read: A is internally related R to B just in case what it is to be A is to be related to B by R. The number 3 is internally related to 2, for example, since what it is to be 3 is to be greater than 2. Contrast this to 'externally' related. Callias may be taller (or greater in height) than Socrates, but they are not internally related, since what it is to be Callias is not connected to his being taller than Socrates. In other words, we can understand what Callias is without having to know anything about Socrates per se. Callias' being taller than Socrates is an example of being externally related. I borrow the distinction (internal versus external) from Wittgenstein, in *Remarks on Colour*, edited by G. E. M. Anscombe, translated by L. McAlister and M. Schaettle (Oxford: Blackwell, 1977), para. 1, p. 2e.

Almost six centuries after Plato, Plotinus developed two theoretical conceptions of matter that are relevant here, which scholars refer to as *inferior*-matter and *material*-matter. Neither purely Platonic nor Aristotelian, they no doubt form a distinction that is a hybrid of the concepts of matter, as Plotinus understood them, developed by Plato and Aristotle. In fact, according to A. H. Armstrong, Porphyry, who Plotinus charged with putting together a written account of his view (which would become the *Enneads*), gave as the original title for Ennead II: 'On the Two Kinds of Matter' (*peri ton duo hylein*).[45] Armstrong casts the distinction as Intelligible-matter versus Sensible-matter.[46] This will be of some interest to us later in Part IV, since according to Descartes several of his critics rely on this very distinction when making their case against his conception of matter (as extension in length, breadth, and depth).[47] As mentioned in the Introduction, Schäfer claims that material-matter is for Plotinus the matter we sense.[48] So, this would align with Armstrong's reading of Sensible-matter. And though I will not deny that the second kind of matter is intelligible, I think Schäfer better captures the distinction when emphasizing that inferior-matter is 'the amorphous *hypokeimenon*', and is for Plotinus matter proper.[49] The distinction on which to focus is not between the matter that we sense versus the matter that we think about (though this distinction is there). Rather, Plotinus' distinction seems to draw a difference between the individuated bodies of the sensible world and the conditions underlying the very possibility of such things so individuated. As I will suggest in Part IV, it is the distinction between what Kant would call the empirical versus the transcendental. Inferior-matter is, as Schäfer contends, 'what we *ex post* identify as the ("in itself") structureless "fabric", which underlies material-matter as we know it of the bodily universe'.[50] As Plotinus puts it, 'What is called "matter" is said to be some sort of "substrate" (*hypokeimenon*) and "receptacle" (*hypodoxein*) of forms (*eidos*).'[51] As I suggested in the Introduction, perhaps a better term than Schäfer's *inferior-matter* here would have been *underlying-matter*. Both Armstrong and McKenna in their translations of the *Enneads* render this 'deeper', underlying-matter *the unlimited* and *the Undelimited* (*to apeiron*), respectively,[52] a concept that we first find in Anaximander. Even so, Plotinus argues against Anaximander's definition of the term if by it was meant 'the infinite'.[53]

Kant

Justifying his claims about the Undelimited, Plotinus writes: 'No doubt there must be a container, as it were a place, to receive what is to enter . . . [and]

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[45] Plotinus, *Enneads*, six volumes, translated by A. H. Armstrong (Cambridge: Harvard University Press, 1966). Armstrong notes this in vol. 2, p. 104.

[46] Plotinus, *Enneads* (Armstrong), pp. 104–105.

[47] AT VII, 212; CSM II, 275.

[48] Schäfer (2004), pp. 266–94.

[49] Ibid., p. 270.

[50] Schäfer (2004), p. 270.

[51] Plotinus, *Enneads* (Armstrong), vol. 2, p. 107.

[52] Plotinus, *The Enneads*, translated by Stephen McKenna, 3rd edn. (London: Faber and Faber, 1962), Second Ennead, Fourth Tractate, p. 116.

[53] Plotinus, *Enneads* (McKenna), p. 109.

the primal necessity . . .[that accounts for] the existence of body, is Matter.'[54] This 'container', or as he casts it in a remark just quoted, this 'receptacle', does more than to suggest that it is akin to the receptacle of Plato. Plotinus continues: 'Matter, in sum, is necessary to quality and to quantity, and, therefore, to body.'[55] Bodies arise as the Soul, or the Reason-Principle as Plotinus calls it, acts on the Undelimited—the Soul acting by 'dividing' the Undelimited. This container, the Undelimited, 'eludes the eye, for it is utterly outside of colour; it is not heard, for it is no sound: it is no flavour or savour for nostrils or palate . . . It is grasped only by a mental process.'[56] Bodies, by contrast, are touched, seen, heard, smelt, and so on. Plato's receptacle and Plotinus' container, as we shall see, seem to be models for Descartes's *res extensa*, which, as I noted in the Introduction, Emily Grosholz identifies as the 'unbounded monolith', standing metaphysically prior to the individuated bodies we observe in the cosmos.[57] Plotinus' remark about the Undelimited—inferior-matter on Schäfer's reading—as something that is not sensed, but is known only by a mental process, should not go unnoticed. For, as we shall also see, this looks to be similar to what Descartes will later say in the Second Meditation about the results of his examination of the nature of the wax qua body (or, of bodies more generally), 'that this perception [of the nature of bodies—i.e., their being extended in length, breadth, and depth] derives not from their being touched or seen but from their being understood'.[58]

Aquinas' account of *number* and *multitude* echoes much of the above, and was a view that was perhaps more directly influential on Descartes's than Plotinus' as laid out specifically in the *Enneads*, though Plotinus' view could certainly have reached Descartes by way of Aquinas (and Augustine). Aquinas says:

> All plurality is a consequence of division. Now division is twofold. The first is material, through the separation of continuous quantity; number, which is a species of quantity, is the result. This numeration is restricted for material things. The second is formal, by contrasted and diverse forms; the result is multitude, not as such included in the physical categories, but ranged with the transcendentals as when reality itself is divided into the one and the many. This kind of multitude applies to supra-material things.[59]

There are two kinds of division. The first, he says, is the division of matter, here taken to be a *continuous* medium, where the divided results (the distinct quantities) are the numbers. The second is the division of the supra-natural, or immaterial, medium. The divided results are the forms, or the transcendentals.

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[54] Plotinus, *Enneads* (McKenna), p. 113. Bracketed addition is mine.

[55] Ibid., p. 113.

[56] Ibid.

[57] Emily Grosholz, *Cartesian Method and the Problem of Reduction* (Oxford: Clarendon Press, 1991), p. 66.

[58] AT VII, 34; CSM II, 22.

[59] *St. Thomas Aquinas: Philosophical Texts*, translated by Thomas Gilby (New York: Oxford University Press, 1960), pp. 158–9. This passage comes from *Summa Theologica*, Ia, xxx, 3.

In Disputations, II *de Veritate*, 4, *ad* 7, Aquinas tells us that, 'The divine essence is [the] universal medium', which, I take it, is the supra-natural, immaterial medium underlying the second kind of division into the system of forms. Alluding to Augustine, Malebranche seems to think something similar in his claiming that the *ideas* are 'limitations' of God's mind, the latter serving as their substance.[60] This second kind of division, of course, is not, strictly speaking, a *division* of the essence of the divine being, for 'God is wholly simple.'[61] The division is connected to having distinct thoughts, for 'we must still address him with a multitude of names'.[62] This second kind of division, Aquinas says, is *formal* (as opposed to material). 'Though we come to know about him [God] through intricate thoughts [read here ideas, forms, or exemplars],' he says, 'we appreciate well enough that the object corresponding to them is simply one.'[63]

Augustine, who would borrow from Plotinus, and whose view would have some influence on Aquinas, entertains important distinctions between the two realms, the spiritual and the material. Ernst Cassirer says that for Augustine 'what wisdom is for souls, number is for bodies'.[64] According to Cassirer, Augustine took the two major divisions of Plato's line to be determined by two distinct epistemic systems: knowledge (*scientia*) and wisdom (*sapientia*). The former was directed at an understanding of temporal (material) things (the highest category of the Visible realm), the latter was directed at an understanding of God (the highest category of the Intelligible realm).[65] If I understand Cassirer's remark correctly, 'wisdom is for souls' is connected to the system of ideas that emerges as the 'division' of the Divine Mind, and 'number is for bodies' is connected to Augustine's thinking that number emerges as a 'division' of the material world (that is, a division into material things). Cassirer says:

> Despite the dualism that Augustine sees between the realms of spirit and matter, then, he maintains that they are linked by a single Reality which manifests itself on the one hand as wisdom and on the other as number.[66]

And, even though mathematics was viewed by Augustine as ultimately a practice leading to God,[67] hence seemingly casting mathematics as a kind of wisdom (*sapientia*) and not a kind of knowledge (*scientia*), the single 'Reality' Cassirer mentions suggests that a study of material things, understood properly, would be a study of the Divine Being. It would not be until Johannes Kepler, says Cassirer, that Augustine's insight here (a mathematical *scientia* as a way of understanding the Divine Being) would be realized.[68]

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[60] Nicolas Malebranche, *The Search After Truth*, translated and edited by Thomas M. Lennon and Paul J. Olscamp (Cambridge: Cambridge University Press, 1980). See, for example, Book III, Part 2, Chapter xi, p. 233, and Book III, Part 2, Chapter xii, p. 237.

[61] Aquinas (*Summa*) Ia, xi. 3.

[62] Aquinas (*Summa*).

[63] *Contra Gentiles*, 36.

[64] Ernst Cassirer, 'Mathematical Mysticism and Mathematical Science', in *Galileo: Man of Science*, Ernan McMullin (ed.) (New York: Basic Books, Inc., 1967), p. 345.

[65] Ibid., pp. 345–6.

[66] Cassirer (1967), p. 345.

[67] See, for example, Augustine's *De Ordine*, translated by Silvano Borruso (South Bend: St. Augustine's Press, 2007), Book I, ix, First Debate, 8.26.

Aquinas seems to be more than acquainted with Plato's position expressed in the *Timaeus*: 'Some have taught that God is the world-soul,' he says, 'while others that he is the formal principle of all things.'[69] Even though he identifies the divine essence as a supra-natural medium (the division of which is the system of forms), Aquinas does not ultimately identify God as the *formal* cause of the cosmos. Neither does he identify God as that material medium (the division of which is the system of numbers)—that is, he does not identify God as the *material* cause of the cosmos. Rather, Aquinas identifies God as the 'essential efficient cause'.[70] In terms of agency, then, he says, 'to speak with accuracy, it is not the hand that acts, but the man through his hand'.[71] Like a man acting through his hand, God acts through these two media—the formal and material media—where the forms are produced through the former, the numbers through the latter.

The two kinds of division, though importantly different, share a common vocabulary, which requires that we be mindful of the differences in meaning that the relevant terms convey. For instance, 'The *unity* convertible with being does not add anything to being, but the *unity* which is the principle of number adds a reality in the category of quantity.'[72] Moreover, '*unity* as the principle of number is contrasted with numerical multitude. But *unity* as convertible with being is contrasted with a kind of multiplicity that spells defect; thus what is single is contrasted with what is in pieces.'[73] The concept of *unity* with respect to number is 1. It is in terms of 1 that all other numbers are defined. Aristotle, for instance, agreeing with certain predecessors, holds that 1 or unity was not a number but the *principle* of number.[74] It would appear that 2 (or the dyad) is the first number. Numbers determined by unity are additive. This much Aquinas adopts.

What appears to be new, however, is Aquinas' position that there are two distinct kinds of unity, and that with respect to the immaterial, its divisions (or divided regions) are *not* additive. The view appears to be lurking in Plotinus' *Enneads* (this is the Divine Matter, or the Base, as he calls it, that serves as medium of pure intellection),[75] but I shall not go into that here. What I will briefly explore, in trying to better sketch a view of the ancients that was in the air during the early modern period, is an interesting connection between Aquinas' second kind of unity and how it is echoed in Galileo's discussion of the infinite versus the discrete (finite quantity), in the *Discoursi*. With respect to infinity, Galileo seems to suggest that familiar mathematical concepts, such as *greater*, *less than*, and *equal*, are unintelligible.[76] He suggests, for example, that infinities are not additive. Discrete quantity by its very definition cannot be infinite (and, unlike infinity, is additive), which is something that Aquinas also says: 'That a natural body cannot be infinite is manifest, for it must have a determinate substantial form, from which determinate accidents result, quantity among them. So every natural body has a determinate quantity.'[77] The infinite is *conceptually* at odds with anything discrete (this is along the lines of Plotinus' rejection of Anaximander's defining matter, the matter underlying bodies, as infinite).[78] Perhaps Aquinas has something like this in mind when talking about unity in this second sense—it is a term connected to infinity-talk, or talk about the transcendentals.

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[68] Cassirer (1967), p. 347.
[69] Aquinas (*Summa*), Ia, iii. 8.
[70] Aquinas (*Summa*).
[71] Ibid.
[72] Ibid., Ia. xi. I, *ad* I.
[73] Ibid., Ia, xi. 2.
[74] Aristotle, *Metaphysics*, Bk. N, 1088a.
[75] Plotinus, *Enneads* (McKenna), p. 107.

?

We might be compelled to say that there is a specific number of angels, for example, even though they are immaterial. So, number would seem to apply to the immaterial. But Aquinas says:

angels

> The number relevant to pure spirits is not that of discrete quantity caused by division of continuous quantity [matter], but the number arising from distinction of forms: in this sense plurality is reckoned among the transcendentals.[79]

use

As we already know, 'the *number* arising from distinction of forms' is not additive—it is not a *discrete quantity* at all, and so is not a number of mathematics.[80] Rather, it expresses the sort of diversity (or plurality) found among the forms, the transcendentals. In light of Galileo's remarks, we might take Aquinas here as thinking of the 'diversity' or 'plurality' as being the sort associated with infinity, and so although one can say that there is a 'multitude' of angels, for instance, given that the multitude is infinite, there is no *number* or *quantity* that *can* be assigned to them. In other words, to think of the infinite as a number—that is, as a number determined by the division of matter—is to make something like a category mistake. This not only goes for angels or created spirits, but for God. 'Taken as a principle of number,' Aquinas says, '*unity* is not to be predicated of God; it applies only to things having existence in matter.'[81]

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[76] Galileo Galilei, *Two New Sciences*, translated by Stillman Drake (Madison: University of Wisconsin Press, 1974), pp. 38–41. Considering the infinity associated with points that 'constitute' a finite line, where a point is said to be an indivisible, Galileo has Salviati say, 'one indivisible joined to another indivisible does not produce a divisible thing' (p. 39), and 'for I consider that the attributes of greater, lesser, and equal do not suit infinites, of which it cannot be said that one is greater, or less than, or equal to, another' (p. 40).

[77] Aquinas (*Summa*), Ia, vii. 3 (418).

[78] Plotinus, *Enneads* (McKenna), p. 109.

[79] Aquinas (*Summa*), Ia., xi. 2.

[80] One wonders about 0. It is not 'additive', so is it not a number? But, 0 is a 'null element' or as it is sometimes put, an 'identity element'. The idea is that with respect to the binary operator +, for example, 0 'nullifies' the operator. That is, it makes it so that the operator has no effect when combining 0 to other numbers. So, we might say, 0 is a number, but one that under some conditions affects the binary operator. With multiplication, for instance, 1 is the null-element. But for all that, 1 is still a number.

[81] Aquinas (*Summa*), Ia. 1. 3, *ad* 2.

The infinite is a transcendental—
unrelated to division of matter

In *de Trinitate*, Aquinas says:

> Some objects of theory . . . depend on matter for their existence. These are subdivided into two groups. Some depend on matter both for their being and for their being understood, for they include sensible material in their definition and cannot be understood apart from it, thus the definition of man involves flesh and bones: with such objects natural science is concerned. But others, though dependent on matter for their existence, can be defined and grasped without reference to sensible matter, thus lines and numbers: with such objects mathematical science is concerned.[82]

Natural science deals with objects that both depend for their existence on sensible matter and cannot be understood apart from sensible matter. Mathematics, on the other hand, deals with objects that depend for their existence on matter, but *can* be understood apart from (or without reference to) sensible matter. About mathematical truths, he says, 'their meaning is independent of sensible matter, though they exist in it . . . Yet their definitive meaning does not abstract from matter, but merely from *sensible* matter.'[83] Let me make this last claim a bit clearer.

About the doing of mathematics, Aquinas says, 'when sensible conditions are set aside [through the process of abstraction] an imaginable residue still remains',[84] which, he indicates, is a kind of *imagined* matter, or as he says elsewhere, 'intelligible matter',[85] or in another place, 'quantity *only*',[86] which he calls a 'mathematical body'[87] as opposed to a natural one. Thus, the definitive meaning of mathematical truths does not abstract from *matter*, but merely from *sensible* matter. The imagined or abstracted material substrate allows the mathematician to acquire knowledge from something that nevertheless depends on sensible matter. Thus, mathematics deals with objects that depend on sensible matter for their existence, but can be understood without reference to it (though apparently reference is made to this imagined or intelligible matter, which seems to be a precursor to Descartes's *extension*, or *body* taken in what he calls the *general sense*).[88]

'Mathematics,' Aquinas says, 'comes midway between physics and metaphysics, and is more certain than either of them.'[89] Physics deals with sensible objects, particular objects of specific quantities (sizes and shapes) that are in motion. The mathematician abstracts from sensible objects, producing an image of an (unchanging) object for study. 'Mathematical objects . . . fall under the senses *and* imagination.'[90] Thus, mathematics 'is more certain than natural science because its reflections are above matter and motion'.[91] It is also more certain than metaphysics, since 'metaphysical objects are more withdrawn from sensation whence our knowledge starts'.[92] Thinking back to Plato's divided line analogy, it would appear that Aquinas is saying that *dianoia*, the mental state connected to doing mathematics, is clearer in some sense than both *pistis* and *noesis*. His division of the sciences into natural, mathematical, and metaphysical would also suggest the following, were we to try to fit it into Plato's scheme:

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[82] Aquinas (*de Trinitate*), v. 1.

[83] Ibid., vi. 2, emphasis mine.

[84] Aquinas (*de Trinitate*).

[85] Ibid., v. 3 (54).

[86] Aquinas (*Summa*), Ia. vii, 3 (418).

[87] Aquinas (*Summa*).

[88] Descartes says, 'the nature of matter, or body considered in general, consists not in its being something which is hard or heavy or coloured, or which affects the senses in any way, but simply in its being something which is extended in length, breadth, and depth'. *Principles*, Part II, art. 4 (AT VIIIA 42; CSM I 224).

[89] Aquinas (*de Trinitate*), vi. 2.

[90] Ibid., vi, 1 (55).

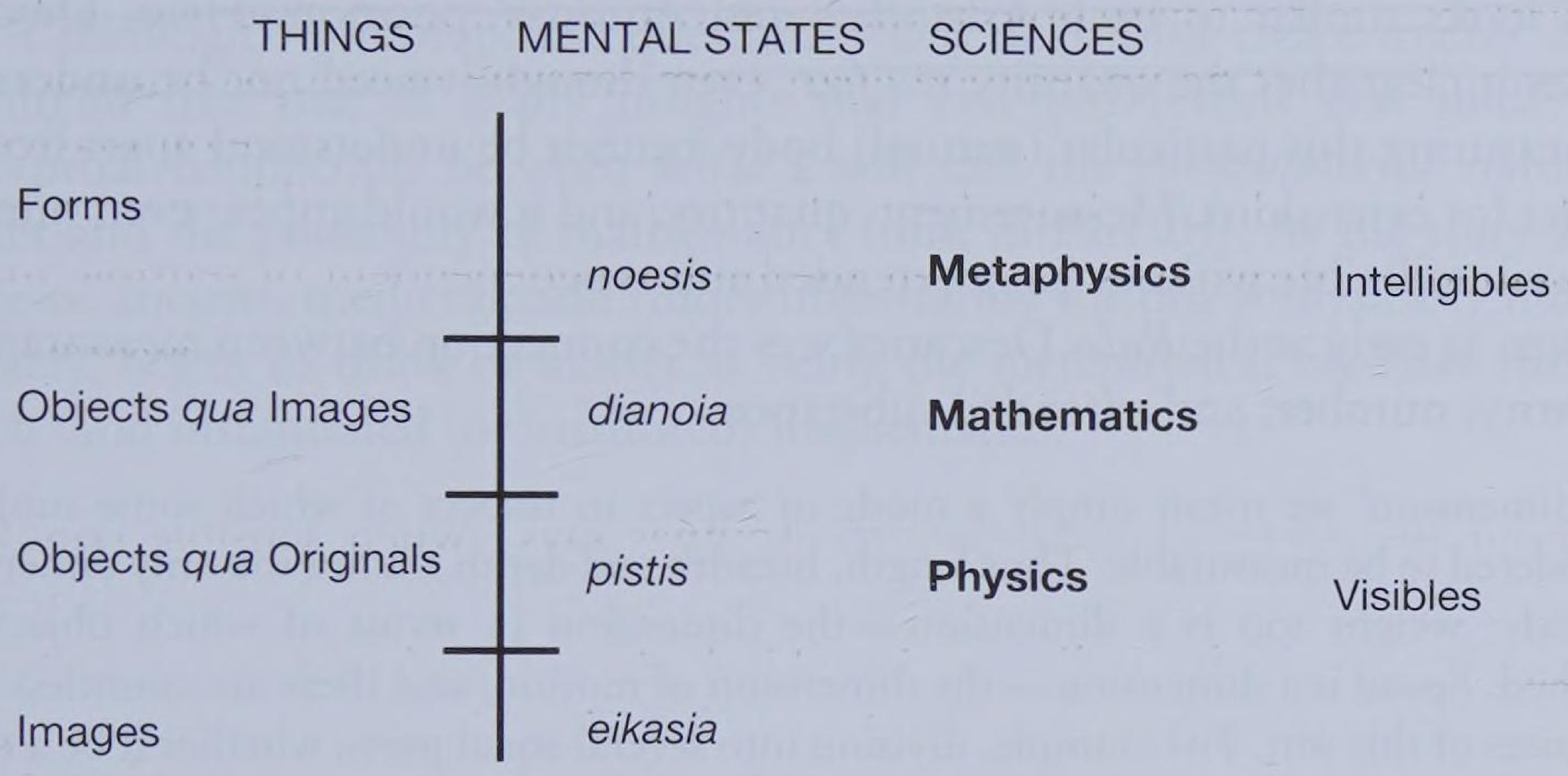

Figure 3.5

If we are right about Aquinas, he would seem to disagree with Plato about the relative clarity associated with the mental states connected to these two classes of objects. And so, he might object to the length of the line subsection representing *noesis* being longer than the one representing *dianoia*, given that length here represented degree of clarity. No doubt Plato (and even Descartes) might see this as a real problem for Aquinas' view.

Aquinas' connecting number or quantity to sensible matter is, not surprisingly, found in Aristotle. In *Metaphysics*, Book N, Aristotle says:

> 'The one' evidently means *a measure*. And in every case there is some underlying thing with a distinct nature of its own, e.g. in the scale a quarter-tone, in spatial magnitude a finger or a foot or something of the sort, in rhythms a beat or a syllable; and similarly in gravity it is a definite weight.[93]

Unity is not only understood as a *measure*, but as a measure *of something*. Crombie notes that this view would again take hold on philosophers of the thirteenth and fourteenth centuries, where physicists interested in applying mathematics 'emphasized the need for the standardization of systems of measurement'.[94]

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[91] Aquinas (*de Trinitate*).
[92] Ibid.
[93] 1087b 33.
[94] Crombie (1961), vol. 1, p. 182.

'Without this habit of thought,' he says, 'mathematical physics would be impossible.'[95] Compare this even to what Descartes says in the *Principles*:

> [W]e can think of the number ten, or the continuous quantity *ten feet*, without attending to this determinate substance. For the concept of the number ten is exactly the same irrespective of whether it is referred to this measurement of *ten feet* or to anything else; and as for the continuous quantity *ten feet*, although this is unintelligible without some extended substance of which it is the quantity, it can be understood apart from this determinate substance.[96]

This looks similar to both Aristotle's and Aquinas' position. Here, Descartes makes it clear that the quantity *ten feet*, even though it need not be understood as measuring this particular (natural) body, cannot be understood apart from *all* matter (or extension). Measurement, quantity, and it would appear even number, are 'unintelligible without *some* extended substance'.

Even as early as the *Rules* Descartes sees the connection between measurement, quantity, number, and extended substance:

> By 'dimension' we mean simply a mode or aspect in respect of which some subject is considered to be measurable. Thus length, breadth and depth are not the only dimensions of body: weight too is a dimension—the dimension in terms of which objects are weighed. Speed is a dimension—the dimension of motion; and there are countless other instances of this sort. For example, division into several equal parts, whether it be a real or merely intellectual division is, strictly speaking, the dimension in terms of which we count things. The mode which gives rise to number is strictly speaking a species of dimension, though there is some difference between the meanings of the two terms . . . Whether dimensions have a real basis is something for the physicists to consider.[97]

I will revisit in more detail the ideas of measurement and quantity (and dimension) in the following chapters. What I want to do here is to flag to the reader what appears to be a significant connection between Aristotle, Aquinas, and Descartes—a connection on which I shall focus later.

Aquinas fails to notice an important implication of his view: given that mathematics requires additive quantity (number determined by unity), and that the immaterial division of reality into forms does not yield additive quantity, it follows that the 'division' of the supra-natural, immaterial medium is not enough to yield the conditions necessary and sufficient for mathematics. A world in which God alone exists is not mathematical, for unity taken as the principle of number does not apply to God. This also will be the case even for a world in which immaterial souls exist with God (I am thinking of the Beatific Vision expressed by Aquinas, or by Dante in his *Paradisio*). Such a world is not mathematical, for the underlying conditions of mathematics is found *only* in 'things having existence in matter'. Thus, given not only what Aquinas says, but arguably what

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[95] Crombie (1961).
[96] *Principles*, Part II, art. 8 (AT VIIIA 44; CSM I 226).
[97] *Rules*, Rule Fourteen (AT X 447-8; CSM I 62–3).

no matter in heaven!

Plato and Aristotle (and even Galileo and Descartes) say, it would seem that in respect to the possibility of mathematics, matter matters. It would seem, then, that an absence of matter entails the unintelligibility of quantity or number, and subsequently the unintelligibility of mathematics.

This is the idea that certain early modern philosophers (like Galileo, Descartes, and Leibniz) seem to be exploring. And, it is to this exploration that we now turn. What we will find is that the 'mathematization' of early modern physics, which I take to be an understanding of the underlying *conceptual* connections between mathematics and the nature of matter, arises in part from an inquiry into a theologico-philosophical question—*why* does God create matter at all? I contend that one of many insights that emerged reveals that underlying conceptual connection between what I will call the *combinatorial* nature of matter and the possibility of mathematics (this, in Part III). As the story above suggests, ancient, medieval, and (more importantly for this study) early modern thinkers, began to think of matter as being the metaphysical medium through which God instantiated (or instituted) mathematics.

PART II

ANALYSIS

4

Analysis and Clarity and Distinctness

In the Second Replies, Descartes reminds his objector of the method used in presenting the *Meditations*. He draws a quick distinction between analysis, which is the method he claims to have used, and synthesis. As Descartes describes them, analysis is in part a style of presentation that shows the reader of the *Meditations* how a thing, or truth in this case, is to be discovered, adding—discovered 'as it were *a priori* (*tanquam a priori*)'.[1] But, it is more than a *style* for presenting philosophical material: it is a *method* of discovery.[2] And, as I shall develop it in what follows, a method of enumerating or dividing that allows its practitioner to determine the basic or atomic categories of a conceptual system. An addition to the French version of the Third Meditation attempts to clarify the analytic method, which Descartes says begins 'only from those notions which I find first of all in my mind and pass gradually to those which I may find later on'.[3] I want to highlight his going on to say that the method of analysis, in moving from what he finds first in his mind to what underwrites these things, requires him to 'divide' his thoughts.[4] The proper division of thought into its simpler elements, we shall see, plays an important role in the development of a mathematical system.

Synthesis, on the other hand, is initially cast as employing 'a long series of definitions, postulates, axioms, theorems and problems', where specific conclusions are drawn logically from these.[5] But, as we shall see in Part III, the method evolves into a kind of combinatorics, where 'conclusions' (or perhaps better, new concepts) arise from the combining of the simpler elements—here the elements are the elements of thought that were discovered by way of analysis. So, more on synthesis in Part III. Part II will focus on analysis.

John Cottingham says that Descartes's use of 'a priori' in his brief description of analysis in the Second Replies is puzzling. 'Descartes's use of the term seems to correspond neither with the modern post-Leibnizian sense . . . nor with the Medieval, Thomist sense.'[6] Although Cottingham does not tell us what he has in mind by 'post-Leibnizian', one assumes he takes a priori to mean 'true independent of sensory experience'. The second possibility that Cottingham gives us, which he attributes to Aquinas, refers to a form of reasoning when one reasons from cause to effect.[7] Cottingham abandons both possible readings, suggesting instead that Descartes is taking a priori to refer to that which a meditator believes to be true prior to engaging in the philosophical investigation as it is laid out in the *Meditations*.[8] And so, supposing that the meditator holds some belief P prior to meditating, which means, if I follow Cottingham correctly, that P is the first of many beliefs to be held in the 'order of discovery', then the *priority* here is attached to P.

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[1] AT VII, 155; CSM II, 110.

[2] Ibid.

[3] AT VII, 36; CSM II, 25. Cottingham's point is made in a footnote in CSM II, p. 25.

[4] AT VII, 155; CSM II, 110.

[5] AT VII, 156; CSM II, 111.

[6] CSM II, fn. 2, p. 110.

Why Cottingham thinks this is not clear. As understood by Descartes and his contemporaries, the aim of the method of analysis is to discover what must be true in order to account for the very *possibility* of P. This, recall from Part I, aligns with what Pappus called the *problematical* version of analysis: 'In the *problematical* kind we assume that which is propounded as if it were known, after which we pass through its successive consequences (*ton heses akalouthon*), taking them as true, up to something admitted.'[9] The 'admitted' something here is a first principle, or self-evident truth, which according to my reading, Descartes would have taken to be a priori. In this context, what the meditator discovers are the a priori conditions underwriting P. In other words, assuming that P and other beliefs like it that the meditator holds prior to meditating are ordinary beliefs (specifically originating in the senses), a priori refers to those conditions that account for the possibility of these ordinary beliefs—it is an account of the possibility of the empirical (or sensible). This reading, it seems to me, better aligns with what he took the aim of the method of analysis to be, which was to discover what underwrites the possibility of what we commonly take to be true. So, contrary to Cottingham's suggestion, which takes the priority to be referring to what the meditator holds prior to meditating—which for discussion's sake is belief P—I am suggesting that by a priori Descartes is referring to whatever it is we discover through meditating that underwrites the possibility of P. To be sure, if Cottingham means that the beliefs underwriting P are prior in 'the order of discovery', as he puts it, then our views are closer than I have laid them out here. However, if so, I am baffled as to why he would deny that a priori is being taken by Descartes in what Cottingham calls the post-Leibnizian sense.

[7] About this, it is helpful to keep in mind that Aquinas connects a priori reasoning to reasoning from cause to effect, and a posteriori reasoning from effect to cause. Thus, if analysis is cast as forwarding some effect (or conclusion) for examination, and moving 'upwards' in search of its cause, on Aquinas' view, this would be to engage in a posteriori reasoning. So, Cottingham rightly finds Descartes's use of a priori here puzzling, if by this he (that is, Descartes) had Aquinas' terminology in mind. I agree with Cottingham that Descartes is not employing Aquinas' terminology. For an excellent discussion of this see also E. M. Curley's 'Analysis in the *Meditations*', in *Essays on Descartes'* Meditations, edited by Amélie Oksenberg Rorty (Berkeley: University of California Press, 1986), pp. 154–5.

[8] Cottingham (CSM II), fn. 2, p. 110.

[9] I quote from Robinson (p. 466), who uses the translation of Thomas Heath, in his *The Thirteen Books of Euclid's Elements*, 2 vols, (New York: Dover Publications, 1956). The passage appears in vol. I, p. 138.

For, understood as I have cast it, a priori is not only in the post-Leibnizian 'ballpark', but squares with the classical take on the concept—which we find in some form or other in Plato, Aristotle, Pappus, and others considered earlier in Part I.

An example of a priori that shows how Descartes employs the concept, I think, is found in his discussions of the conceptual relation between *shape* and *extension*. In the *Principles*, he says:

> Everything else which can be attributed to body [e.g., shape and size] presupposes extension, and is merely a mode of the extended thing . . . For example, shape is unintelligible except in an extended thing.[10]

The thing to focus on is the remark about shape's being unintelligible apart from extension. What we discover is that *necessarily* if one conceives of a shaped thing, one is also conceiving of an extended thing. So, let P be '*A* is shaped', where *A* is an ordinary object of the senses. In discovering that '*A* is extended' is a necessary condition for P, what we discover is an a priori condition for something we originally held. The necessity here expresses the sense in which extension is an a priori condition of the very possibility of being shaped.

There are other texts. In the Second Meditation, for example, Descartes famously analyses his sensory perception of a piece of wax.[11] Supposing that *A* is that piece of wax, what the analysis reveals is that, in being a necessary condition for being shaped (sized, etc.), extension is the nature or essence of *A*. In fact, extension is the nature of anything that is shaped (sized, etc.). How is this determined? In seeing that extension is a necessary condition for shape, how does this establish that it is a priori? In line with what he says in the above *Principles* passage, we determine its nature by discovering that we cannot even conceive of a shaped thing, or more generally, a *body*, without also conceiving of it as being extended. How is this a priori?

That extension underlies the very intelligibility or conceivability of *body* is not something we learn from our sensory experience of bodies. For, as in the case of the piece of wax, the only qualities we sense are colours, sounds, tastes, and so on. And, with respect to shape, size, etc., when sensing the wax, *this* piece of wax, here and now, it is presented as having a *particular* shape, a *particular* size, and so on. At other times, for example, when we return to observe the wax after it has been sitting on the warm hearth, we may very well be presented (in sensory experience) with an object whose *particular* shape, size, colour, etc., are quite different from that of the object presented before.[12] For all we know, Descartes hints, given *only* the comparison of the two moments of sensory perception (where the comparison itself cannot be done via the sensory faculty alone—we would require a faculty of memory, for example), we are being presented with two distinct objects. There is absolutely nothing *in* our sensory experiences that ties the presented objects together so that we are justified in believing that these are two sensory experiences of the *same* object.

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[10] AT VIIIA, 25; CSM I, 210.

[11] AT VII, 30ff; CSM II, 20ff.

[12] AT VII, 30; CSM II, 20.

Descartes turns to the imagination as a possible origin for our knowledge of the nature of that piece of wax, but more importantly, of the nature of body. He notes that we can imagine the same object—here, the same piece of wax—having various shapes over time.[13] But how many can I imagine? Certainly not an infinite number, for to imagine this would take nothing short of an infinite amount of time. And, although I can imagine the wax's having more shapes than I have actually seen (in observing the wax via my senses), there is a limit to what I can imagine. The limit is not only in respect to the number of shapes, but what we learn in the Sixth Meditation is that, after some effort, shapes in the imagination become, for lack of a better term, indistinguishable.[14] The image of a thousand-sided figure, for example, appears no different from a thousand-and-twenty-six-sided figure. But, the point seems to be that even though I can imagine the wax's having some large number of different shapes, the *fact* is that I cannot imagine all the possible shapes it can have.[15] And yet, the very thing that I believe about the wax (or any body for that matter) is that it can take on an infinite number of different shapes. Where does this belief originate? Not in the senses, and not in the imagination. The belief has its origin in the faculty of reason, and is a belief that is true independently of any particular instance of sensory experience—this despite the fact that our discovery of it began with considering the limits of sensory experience.[16] The claim is, in other words, *a priori*. And so, for Descartes, *what underlies all possible shaped things is extension* and *the nature of body is extension* are examples of claims that are *a priori*, and they are discovered by way of the method of analysis.[17]

In his look at Descartes and analysis, Edwin Curley rightly makes the connection to Plato.[18] Curley's description of the 'Platonic procedure' that he says Descartes employs, however, is problematic from my point of view. For, he seems to mistake the dialectical *style* of presentation with the *method* of analysis. Curley says, now thinking of Plato, that Descartes's method of analysis begins 'with a conjecture, considering what can be said against that conjecture, and then revising the conjecture in whatever ways the objections suggest. The initial conjecture may be (typically, will be) a false start, in the sense that it will ultimately be rejected in the form in which it is first proposed.'[19] But, as Sayre pointed out (as discussed in Part I), this does not describe Plato's method of analysis, but better describes Plato's earlier method of hypothesis. The method of analysis is a method of collection and division, which, according to Sayre and Cornford, *replaced* the method of hypothesis. To be sure, Descartes does employ an internal dialogue (or dialectic) as part of the narrative style of the *Meditations*, and it certainly could be taken as the style of an analytic presentation, but this style of presentation is not the *method* of analysis.

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[13] AT VII, 31; CSM II, 20–21.

[14] AT VII, 72; CSM II, 50.

[15] AT VII, 31; CSM II, 21.

[16] AT VII, 31; CSM II, 21.

[17] Descartes refers to the investigation as a 'purely mental scrutiny' (AT VII, 31; CSM II, 21).

[18] Curley (1986), p. 157.

[19] Ibid.

Curley mentions *enumerating* and *enumeration* only a few times, and in these contexts he seems to mean by it something like *list making* for the former and *a list* for the latter. This is understandable, of course, since these are standard ways of taking these terms. For instance, Curley says, 'After having explained once what a thinking thing is by enumerating various synonyms—*mens*, *animus*, *intellectus*, and *ratio*—Descartes returns to the topic when he later enumerates the various activities which exemplify thought: doubting, understanding, affirming, denying.'[20] These lists are, according to Curley, examples of enumerations. Another instance is found in Curley's explanation of Descartes's Third Meditation proof for the existence of God. There, he characterizes Descartes's 'enumerating the attributes our idea of God represents him as having' as nothing more than the providing of two 'lists'.[21] He says, 'I take it that he [Descartes] really regards the enumeration of attributes as not ultimately a satisfactory way of clarifying this concept.'[22] I will show in what follows that this sort of reading makes three crucial mistakes. First, it fails to see that Descartes has a highly technical sense of 'enumerating' at work in the *Rules*, which is found at work in the *Meditations*—Curley, for example, wrongly taking it in the ordinary sense of list making. Second, it fails to see the connection between enumerating and the Cartesian concepts of clarity and distinctness. Third, it fails to see that enumerating, understood in its technical sense, *is* the method of analysis.

In what follows, I argue that Descartes's method for becoming clear and distinct—that is, for producing clear and distinct perception—is the method of collection and division, the method of analysis introduced in Part I. Towards the close of this discussion, I will show how the result of analysis—a *partition* of ideas—is the Cartesian scheme of clear and distinct ideas and, as it turns out, a result in part that underwrites the conditions necessary for the concepts of *quantity* and *measurement*. By the end of Part II, it should be clear to the reader what part analysis played in the development of a mathematical physics. I will begin with some of the very first mentions of clarity and distinctness by Descartes, in what I think are technical senses, which can be found in early works such as the *Geometry* and the *Rules*, and move then in the next two chapters to a brief development of a general theory of clarity and distinctness, principally as it is developed in the *Rules*.

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[20] Curley (1986), p. 160.

[21] Ibid., p. 168.

[22] Ibid.

A BRIEF LOOK AT THE TEXTS

Although the *Geometry* was published in 1637 as part of the famous *Discourse On Method*, there is good evidence suggesting that it was a work in progress as early as 1619.[23] As we shall see below, there is also evidence suggesting that Descartes was very likely working on the *Rules* about this time. Some interpretive remarks about the concepts of clarity and distinctness, and about the terms used by Descartes to denote them, are in order before moving on to the specifics of the earlier texts.

In his recent book, *The Plain Truth*, Thomas Lennon criticizes the standard English translation and subsequent interpretative reading of Descartes that have emerged over recent decades.[24] He argues:

> [T]he now-standard translation of Descartes, by Cottingham et al., does us a disservice in this instance by repeatedly translating Descartes's adverbial statements of his alleged criterion in substantival terms. Thus, in the most important text of all, the truth-rule paragraph of Meditations III, where Descartes says that the cogito would not be enough if something that he clearly and distinctly perceived should be false, Cottingham et al. have him perceiving *with* clarity and distinctness. Descartes himself could not have expressed himself in this way because there is no Latin for it. To be sure, there is *claritas*, but there is no substantive that is cognate with the adjective *distinctus* or the adverb *distincte*. The nouns *distinctio* and *distinctus* mean distinction, not distinctness.[25]

So, the Latin terms *clare* and *distincte* are to be rendered *clearly* and *distinctly* in English, strictly as adverbs—as in the phrase *clearly and distinctly perceiving x*—and not as *clarity* and *distinctness*, as in the phrase *perceiving x with clarity and distinctness*. According to Lennon, the trouble with the substantival rendering of these (adverbial) concepts is that they have misled commentators to think of clarity and distinctness as criteria for truth, which has emerged as the now-standard view. Lennon rejects the now-standard view, denying that clarity and distinctness are criteria for Descartes.[26]

We find Descartes using the French '*nettement*' in the *Discourse*, which is usually translated as the adverb *clearly*. He writes, '*concevoir plus netement* [sic] *&* *plus distinctement ses objets*',[27] which can be translated, 'to conceive more clearly and more distinctly its objects'. The French '*nettement*' (which in the Adam and Tannery volume is misspelled in this instance) is paired with '*distinctement*'. In the

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[23] Described in a number of early letters to Beeckman, Descartes actually mentions the work's *name* in a letter dated 23 April 1619 (AT X, 162; CSMK III, 4).

[24] Thomas Lennon, *The Plain Truth: Descartes, Huet and Skepticism* (Leiden: Brill, 2008), pp. 168–72.

[25] Lennon (2008), pp. 169–70. The emphasis of the italicized *with* in this passage is mine.

[26] Lennon (2008) informs his reader of this on p. 55. The argument can be found on pp. 170–72.

[27] AT VI, 21.

Geometry, published along with the *Discourse*, but written in some form prior to the *Discourse*, Descartes employs language similar to that found in the *Discourse*. He writes, for example, '*conçoive aussy nettement & aussy distinctement*',[28] which can be translated, 'to conceive as clearly and as distinctly'. Again, '*nettement*' is paired with '*distinctement*'. To be sure, this could be explained by taking Descartes to have reworked the language of the *Geometry* to align with the language of the *Discourse*. But, as we will see in what follows, we find a view of clarity and distinctness worked out in some detail in the *Rules* that is consistent with what we find in both the *Geometry* and the *Discourse*. No one thinks that Descartes went back to rework the *Rules* to align with the *Discourse*. So, any similarities between the language of the *Rules* and of the *Geometry* can, it seems to me, be attributed to the view Descartes was developing early in his career (some time between 1619 and 1628).

Among those works intended for publication, it is not until the *Meditations* that we begin to find Descartes using the familiar '*clare*',[29] the Latin adverb for 'clearly'. Even so, Descartes does use the Latin '*clarus*',[30] the adjective for 'clear'. And, in the French version of the *Meditations* (published after the Latin version), which Descartes approved, both forms of expression are repeated. What is more, as was noted above, Descartes pairs the adverbs *clearly* and *distinctly*, and the adjectives *clear* and *distinct*. For example, we find the adverbial phrasing, '*conçoive clairement & distinctement* (to conceive clearly and distinctly)',[31] and we find the adjectival phrasing, '*idée claires & distinctes* (clear and distinct ideas)'.[32] The French '*nettement*' appears to have been dropped in favour of '*clairement*'.[33] Not only does Descartes continue to pair the Latin adverbs *clare* and *distincte* (and the adjectives *clara* and *distincta*), but he makes it a point on several occasions to contrast *clara* with *obscura*, and *distincta* with *confusa*.[34] Thus, the pair '*clarae & distinctae perceptionis*' (of clear and distinct perception), for example, is contrasted with the pair '*obscurae & confusae (perceptionis)*'.[35]

If I understand Lennon correctly, the reason that clarity and distinctness are not criteria (of truth) for Descartes is because they are not features to be found *in* ideas. To clearly and to distinctly perceive, argues Lennon, is to simply succeed at seeing, or arriving at, the truth.[36] *Perceiving*, he says, is what Ryle called an *achievement word*—where to clearly and to distinctly perceive is, on Lennon's view, Descartes's way of emphasizing the success of perceiving.[37] Alternatively, Lennon puts it, 'Perceiving the truth clearly and distinctly is not some mysterious additional episode [to perception *simpliciter*]. It is just perceiving the truth.'[38] Thus, on Lennon's view, in recasting the phrase *clearly and distinctly perceiving* as *perceiving with clarity and distinctness*, Cottingham et al. prompt us to think of clarity and distinctness as substantives (in this case, as items to be found in, or accompanied by, ideas), which has misled those who advocate for the now-standard view.

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[28] AT VI, 392. [29] AT VII, 43. [30] AT VII, 44. [31] AT IX, 34.

[32] AT IX, 35.

[33] The French '*nettement*' is rooted in the Latin '*nitidus*', which means 'bright or shining'. The Latin '*clarus*' means 'clear, bright, illustrious'. So, it seems reasonable to think that as *nitidus* and *clarus* are synonyms, so are *nettement* and *clairement*. Descartes will sometimes use the Latin '*purus*' instead of '*clarus*', which is 'in the ballpark', meaning 'pure, spotless, without stain'.

[34] See, for example, AT VII, 63 and AT VII, 164. [35] AT VII, 164.

[36] Lennon (2008), p. 171. [37] Ibid.

There are passages, I believe, that count against Lennon's claim. In the Third Meditation, for example, Descartes writes, '*Ex iis verò quae in ideis rerum corporalium clara & distincta sunt*'[39] which Cottingham translates as, 'With regard to the clear and distinct elements in my ideas of corporeal things'.[40] Here, Cottingham rightly attributes *clara* and *distincta* to *iis . . . quae in ideis* (those [things] which [are] in ideas). Now, that those 'things' are *elements*—that is, elements in some metaphysical or ontological sense—is unclear. I will later argue that these are in fact elements for Descartes, which in the *Rules* he calls 'simple natures'. But, Cottingham nowhere explains why *he* translates the '*iis . . . quae in ideis*' phrase so as to pick out *elements* in ideas. Even so, whatever *clara* and *distincta* are, they are not adverbial expressions, and so are not telling us about the act of ideating or perceiving, but are attributed to *that which is in* the ideas. To be sure, as Lennon notes, when we find Descartes attributing *clare* and *distincte* (adverbs) to the act of ideating or perceiving, we should render them *clearly* and *distinctly*, but my point is that Descartes will also sometimes attribute *clara* and *distincta* (adjectives) to his ideas or to the objects that are *in* his ideas.

There are other passages that also seem to speak against Lennon's reading. For example, in the Second Replies, Descartes says that seekers of truth could never go wrong were they to rely 'solely on the intellect in their quest for clarity in their perceptions (*iis qui claritatem suae perceptionis*)'.[41] The quest is for clarity (*claritatem*)—for *that which is clear*, something that is located *in* perception. Later, again in the Second Replies, Descartes tells Mersenne that the reliability of the intellect even grants an investigator a certain assurance about the nature of God, such that the investigator can truly 'assert that we have examined his nature with sufficient clarity (*satis clare ipsam investigasse*).'[42] These uses of *claritatem* and *clare* by Descartes, it seems to me, open the door to Cottingham's translating them, as he does in the above cases, as the substantive *clarity*. So, it appears that Descartes does sometimes suggest that clarity and distinctness are to be found *in* ideas—or, at the very least, as features that indicate the presence of truth in our ideas. And so, the possibility of their being criteria is rekindled. Is there any way, then, to explain, in a way that might satisfy Lennon, how Descartes might be taking *clara* and *distincta* in such cases—that is, as criteria of truth?

[38] Lennon (2008), p. 172.

[39] AT VII, 44.

[40] CSM II, 30.

[41] AT VII, 147; CSM II 104.

[42] AT VII, 152; CSM II 108.

Let me begin an answer to this question by introducing the following analogy: the case of an eye examination.[43] Suppose that Jones goes to the optometrist to acquire a new prescription for a pair of eyeglasses. At some point in the examination, the optometrist tries to determine the proper thickness of lens for Jones' new glasses. The optometrist asks Jones to look through various lenses while looking at a black dot on the wall directly in front of him. As the optometrist places different lenses before Jones' eyes (so that he is *looking through* them), Jones reports to the optometrist when a lens brings the dot into focus, and in particular, when a lens brings the dot into perfect focus. By 'perfect focus' it is meant that the edge of the black dot is presented in its sharpest contrast to the surrounding white field of the wall. So, by way of the dot, the *object* of vision, Jones can help the doctor come to know when the *act* of vision is clearest for him.

Apply this to objects of ideas and to acts of ideating. For Descartes, there may be a sense in which our noting certain things about *objects* presented in our ideas is informative about the *acts* of ideating. If the above analogy holds, casting clarity and distinctness in terms of object-talk (as substantives or adjectives) is not incompatible with casting them in terms of act-talk (as adverbs). To say, then, that I see the dot *with clarity* (or that my sight of the dot *is clear*), I can be taken as saying that I *clearly see* the dot. As in the eye-examination case, Descartes seems to hold that there are certain features found *in* our ideas (the features being *clara* and *distincta*) that correlate to certain features attributed to the act of ideating (the features here being *clare* and *distincte*). What will be important for my account (to be given below) is that they need not be two distinct *types* of feature, but could be two different ways of talking about the same features. But I am getting ahead of myself, for all I wish to establish at this point, contra Lennon, is that there are passages in which Descartes treats clarity and distinctness as substantives (or as items to be found in ideas), and that in such cases he can be taken to be suggesting that they are items that serve as criteria for (the presence of) truth.[44]

Consider one last remark about Descartes's use of Latin. In the *Rules* he says:

> I wish to point out here that I am paying no attention to the way these terms have lately been used in the Schools. For it would be very difficult for me to employ the same terminology, when my own views are profoundly different. I shall take account only of the meanings in Latin of individual words and, when appropriate words are lacking, I shall use what seem the most suitable words, adapting them to my own meaning.[45]

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[43] Nelson Pike introduced this sort of case to me in a discussion.

[44] Although I will not address it here, I will later introduce Descartes's remarks concerning the fact that our will is *compelled* to affirm an idea's truth whenever the idea is clear and distinct. This, it seems to me, is yet another indicator of truth. At this point in the discussion, then, I simply wish to establish the possibility of Descartes's treating *clarity*, for example, as a substantive, in response to Lennon's claim that such would have been impossible for Descartes to have done.

[45] AT X, 369; CSM I, 14.

In Rule Three, for example, in which he introduces the technical concept of intuition, Descartes uses for what is translated as 'clear', as in 'clear and attentive mind', the Latin '*purae*', in the phrase '*mentis purae & attentae*'.[46] This is a rather important technical matter in the *Rules*. Note that he is not using '*clarae*', the term he seems to prefer later. Even so, '*purus*' and '*clarus*' are synonyms, and so I see no obvious troubles here. The above *Rules* passage is not unique on this front. In a letter to Beeckman, for example, Descartes refers to a work written by the famous logician Raymond Lull as the '*Ars Brevis*'.[47] A month later, in another letter to Beeckman, Descartes again refers to this very same work, but now calls it the '*Ars Parva*'.[48] '*Brevis*' and '*Parva*' are synonyms—but they are different Latin words. My point is that Descartes does not seem concerned over *which* Latin words he uses, so long as they convey his meaning. So, contrary to Lennon, I do not believe that we can deny Descartes a system of technical terms that denotes his criteria for truth—terms which denote technical philosophical concepts—based simply on the fact that he failed to employ one set of Latin (or French) terms in those discussions, or simply on a propensity of English translators to render '*clare*' and '*distincte*' as the substantives *clarity* and *distinctness*.

Now, in the *Geometry*, Descartes develops a method by which all geometrical problems can be solved, regardless of their number or difficulty. As part of this method he develops in some detail a notation for expressing mathematical formulae.[49] In these discussions he emphasizes the role that equations play in his method. To begin, he says, once we know the kind of problem with which we are dealing, for example, whether it is a plane or conical problem, we can then sort out the 'parts' of the problem into two classes—into the known and the unknown.[50] The known parts will be represented by numbers or certain letters of the alphabet, a, b, c, and so on, and the unknown parts will be represented by other letters, for example, x, y, and z. In making explicit the relationships these parts have to one another, we form or construct an algebraic equation.

In general, the strategy for solving equations is to *reduce* the unknown parts to *one* unknown part, and then to move all of the known parts to one side of the equation and the unknown part to the other. When this is completed, Descartes says, we will know the unknown quantity in terms of the known.[51] Take, for example, the equation $ax + b = c$, where x is the unknown quantity and a, b, and c are known. To solve this equation, one way to go about it is to first move x to one side of the equation and all the other 'parts' to the other. We might do this by first moving b over to the right side by subtracting it from both sides. Thus:

$$ax = c - b. \tag{1}$$

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[46] AT X, 368. [47] AT X, 156; CSMK III, 2. [48] AT X, 164; CSMK III, 4.

[49] Here I cite, along with the AT numbers, David Eugene Smith's and Marcia Lantham's translation of the *Geometry* (New York: Dover Publications, 1954). I shall abbreviate Latham and Smith as 'LS.' So, this passage is: AT VI, 371; LS, 5.

[50] AT VI, 372; LS, 6. [51] AT VI, 372–3; LS, 9.

We can then move *a* to the right side by dividing both sides by *a*. And so:

$$x = \frac{c - b}{a}. \tag{2}$$

The unknown quantity, *x*, is now known in terms of the known quantities. The equation makes explicit the relationships that the known quantities have to one another. In our example, *c* is subtracted by *b*, where the remaining quantity is divided by *a*. The quantity expressed by the known parts *is x*, the unknown.

The method in the *Geometry* demonstrates that there is a way to translate geometrical problems into algebraic ones, and algebraic problems into geometrical ones. Ultimately, Descartes intended the method to be applied in the sciences. Motion is conceived in geometrical terms and translated into algebraic terms. Descartes shows how the translation works by showing how a quadratic can be solved both algebraically and geometrically. He says:

> For example, if I have $z^2 = az + b^2$, I construct a right triangle NLM with one side LM, equal to *b*, the square root of the known quantity b^2, and the other side, LN, equal to 1/2 *a*, that is, to half the other known quantity which was multiplied by *z*, which I supposed to be the unknown line. Then prolonging MN, the hypotenuse of this triangle, to O, so that NO is equal to NL, the whole line OM is the required line *z*.[52]

The geometrical drawing that this passage describes is:

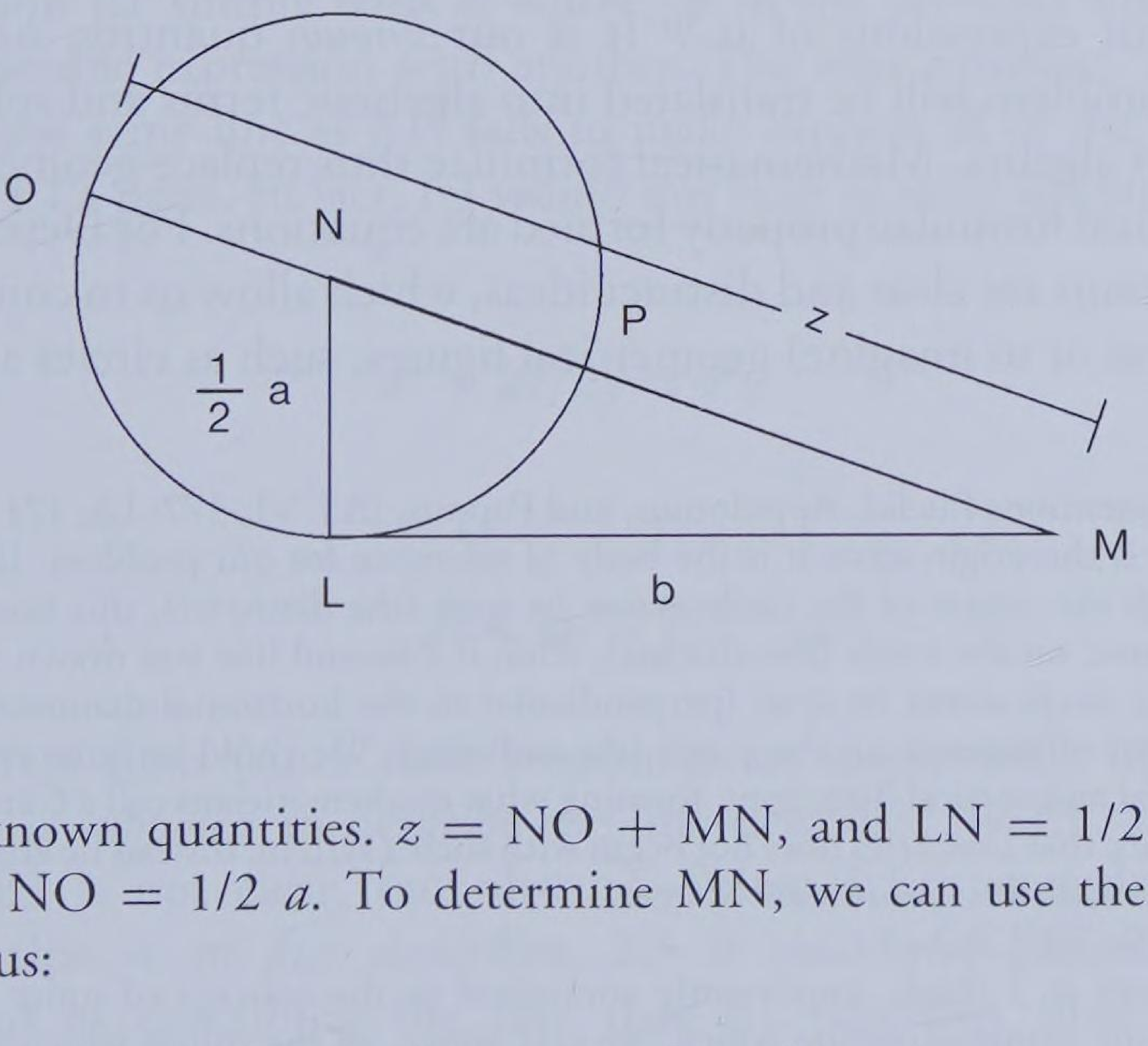

Figure 4.1

a and *b* are known quantities. *z* = NO + MN, and LN = 1/2 *a*. Since LN = NO, then NO = 1/2 *a*. To determine MN, we can use the Pythagorean Theorem. Thus:

$$MN^2 = (1/2\ a)^2 + b^2. \tag{1}$$

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[52] AT VI, 374–5; LS, 13.

Simplifying this, we get:

$$\mathrm{MN}^2 = 1/4\, a^2 + b^2. \tag{2}$$

And, solving for MN, we get:

$$\mathrm{MN} = \sqrt{1/4\, a^2 + b^2}. \tag{3}$$

Hence, if $z = \mathrm{NO} + \mathrm{MN}$:

$$z = 1/2\, a + \sqrt{1/4\, a^2 + b^2}. \tag{4}$$

Descartes's solutions to difficult mathematical problems become more sophisticated as the *Geometry* unfolds. According to Descartes, the method allows him to solve problems that mathematicians before him could not solve.[53] At the risk of oversimplifying the method, the procedure goes something like this: first one must establish a point of reference in the problem (geometrically conceived or 'drawn') to which everything in the problem will be related. Descartes calls this the 'origin'. Once the origin has been established, one must determine a magnitude of length. Although this length is arbitrary, once it is fixed, it will serve as the unit of measure, the 'absolute' in the problem in terms of which all distances shall be measured.[54] Descartes calls this fixed distance 'unity', since it is the measure that *unites* the problem.[55] All distances will have unity in common, that is, they will be various expressions of it.[56] It is our *known* quantity: *unity* = 1. The geometrical problem will be translated into algebraic terms and solved according to the rules of algebra. Mathematical formulae thus replace geometrical images.

Mathematical formulae properly formed are equations. For Descartes, properly formed equations are clear and distinct ideas, which allow us to conceive (and not simply to sense or to imagine) geometrical figures, such as circles and triangles.[57]

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[53] Descartes mentions Euclid, Appolonius, and Pappus. (AT VI, 377; LS, 17)

[54] The circle is the origin since it is the body of reference for our problem. If a horizontal line is drawn through the centre of the circle across its span (the diameter), this line would represent one unit of distance on the x-axis (the abscissa). And, if a second line was drawn vertically through the centre of the circle across its span (perpendicular to the horizontal diameter) this line would represent one unit of distance on the y-axis (the ordinate). We could imagine repetitions of these lines in horizontal and vertical directions, forming what mathematicians call a Cartesian Coordinate System. But notice that Descartes does not begin with such a system, instead he arbitrarily constructs something that entails this Coordinate System.

[55] AT VI, 369–70; LS, 2–3.

[56] This concept is, I think, importantly connected to the concept of unity developed in the *Rules*: 'Unity is the common nature which, we said above, all the things which we are comparing must participate in equally. If no determinate unit is specified in the problem, we may adopt as unit either one of the magnitudes already given or any other magnitude, and this will be the common measure of all the others' (AT X, 449–50; CSM I, 63).

[57] AT VI, 392; LS, 47.

As noted earlier, this is one of the first instances in which we find Descartes using the terms 'clearly' and 'distinctly' in what I will now show are terms with technical senses. By 'technical senses', I mean that they are terms that are doing specific systematic philosophical work. If I follow Descartes's discussion in this section of the *Geometry* correctly, an equation is *clear* because the *relations* between all of the *parts* or *elements* of the equation are explicitly shown.[58] Nothing is obscured from view. An equation is *obscure*, then, whenever it fails to make explicit all of the relations that hold between the equation's parts. Although he does not give any examples of such equations, I take it that the following would meet the above description.

First, consider the equation:

$$a^2 + ax/2y = y^2. \tag{E1}$$

Equation E1 is clear, since it explicitly shows the relations that hold between its elemental parts. Now, if we were to replace x and y with, say, z and w, the new equation (call it E2)—that is, its structure—would be identical to E1:

$$a^2 + ax/2y = y^2. \tag{E1}$$

$$a^2 + az/2w = w^2. \tag{E2}$$

And since E1 is clear and distinct, so is E2. But, suppose that we replaced $x/2y$ in equation E1 simply with z, where, as in the previous substitution, we replace one variable expression with another. The new equation, E3, (regarded as expressing the same line as E1) fails to make explicit all of the relations that hold between E1's parts. In fact, E3 would not map us onto the same line as E1. That is,

$$a^2 + ax/2y = y^2, \tag{E1}$$

and,

$$a^2 + az = y^2, \tag{E3}$$

describe different lines (since their structures are importantly different). But, given that E3 is *supposed* to describe the line described by E1, and E1 is said to be clear, E3 is not clear, but obscure. When E3 is regarded as describing the line that it *in fact* describes, E3 is clear and distinct. It is only when regarded as describing the line that E1 describes that the equation is obscure. The z element, we might say, *obscures* from view the essential relationship expressed between a, x, 2, and y. *Clarity* and *obscurity* in this sense are opposites.

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[58] See also what he says at AT VI, 473; LS, 215.

An equation is *distinct* whenever it describes one and only one line. It separates this line out from all others. For any value assigned to x, the equation will give us a value y such that the ordered pair $<x, y>$ will describe a point on a line. If the equation were to describe more than one line, it would be confused (*confusio*). Such an equation would 'mix together' ('*con*' = *with* or *together*, '*fusio*' = *fuse* or *mix*) equations that describe otherwise distinct lines. An equation that describes a hyperbola, when the hyperbola is regarded in a certain way, is an example of an equation that confuses (or mixes together) the equations of two distinct (though related) lines.[59] *Distinctness* and *confusion* are thus opposites.

On this account, equations can be clear and distinct, clear and confused, or obscure and confused. They cannot be obscure and distinct, since such an equation would both describe and fail to describe one and only one line—which is impossible.[60] The following are examples of the three possible kinds of equations:

Clear and distinct: $$a^2 + ax/2y = y^2. \tag{E1}$$

Clear and confused: $$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1. \tag{E4}$$

Obscure and confused: $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{E5}$$

The obscure and confused equation, E5, is so because (1) it does not explicitly show the relation that x and y have to the hyperbola's centre $<h, k>$, whereas the clear and confused equation E4 does make this relation explicit (which is why E4 is clear), and (2) the equations E5 and E4 each describes at least two distinct lines, which is why they are confused. Whereas Descartes appears to be employing the concepts of clarity and distinctness in the *Geometry* without any detailed discussion of them, it is in the *Rules* that he works out the theoretical underpinnings of these concepts. Let me now turn to laying some of this out.

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[59] I suspect that were we to regard the hyperbola as a single conic section, then its equation would be considered clear and distinct. My suspicion is based upon certain remarks Descartes makes about regarding a triangle inscribed in a semicircle as a single geometrical figure. (AT VII, 224–7; CSM II, 157–9) However, it appears that even the equation for a circle can be regarded as confused, since it can be conceived as describing two semicircles.

[60] Suppose that we take $a^2 + az = y^2$ to describe the line described by $a^2 + ax/2y = y^2$. As was argued, given this supposition, the former equation is obscure. But, in this case, it appears the equation describes two lines: the line described by $a^2 + az = y^2$ and the line described $a^2 + ax/2y = y^2$. To say in this context that the former equation is distinct is to say that by this equation describes one, and only one, line. Hence, in this context, to say that the former equation is obscure and distinct implies that the equation describes and does not describe one line.

5

A General Theory of Clarity and Distinctness

Clarity and distinctness are among the most important concepts in Cartesian epistemology. A comparison of views that have emerged from commentators would appear to show that there is no single theory of clarity and distinctness, but several diverse theories. Most notable are those worked out by Curley, Alanen, and Nelson.[1] It seems to me that Lennon, in denying that clarity and distinctness are criteria, would deny that there is any theory to be found in the texts. I contend that there is one general theory that can account for the diverse theories that Curley, Alanen, and Nelson see in the texts. The diverse theories stand to the general theory as diverse species stand to a common genus—hence the name *general theory*. What is more, in holding that there is a general theory, and given that I have Lennon's view right, I would reject any view, including Lennon's, that claimed that there was no theory to be found in the texts.

As in the *Geometry* there are steps in the *Rules* that must be taken in order to arrive at solutions to problems. These steps constitute Descartes's method. In the *Rules* they, not surprisingly, are referred to as 'rules'. The *Regulae ad Directionem Ingenii*—the *Rules for the Direction of the Mind*—which is the full title of the work, is believed to have reached its end in 1628.[2] Daniel Garber claims that Descartes probably began work on the *Rules* as early as 1619.[3] My own assessment, stated earlier, agrees with Garber's. It was not published during Descartes's lifetime. It was supposed to have contained three sets of rules, each set containing twelve. According to Murdoch, who translated the text into English, the first set is 'concerned with simple propositions and the two cognitive operations by means of which they are to be known, intuition and deduction'.[4]

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[1] I discuss each in my, 'A General Theory of Cartesian Clarity and Distinctness Based on the Theory of Enumeration in the *Rules*', *Dialogue*, vol. XL, no. 2 (spring 2001), pp. 279–310. For a closer look at these views, see Curley (1986); Lilli Alanen, 'Sensory Ideas, Objective Reality, and Material Falsity', in *Reason, Will, and Sensation*, edited by John Cottingham (Oxford: Clarendon Press, 1994); and Alan Nelson, 'The Falsity in Sensory Ideas: Descartes and Arnauld', in *Interpreting Arnauld*, edited by Elmar Kremer (Toronto: Toronto University Press, 1996).

[2] Adam and Tannery (see Descartes, 1983), at AT X, 351–7, refer to the work as it is found (in part) in the pages of nine notebooks, as *Traité des Regels utiles & claires pour la direction de l'sprit* (Treatise on the Useful & Clear Rules for the Direction of the Mind). I mention this because some scholars prefer to emphasize the work's alternate title term, *ingenni* (meaning innate or native intelligence).

[3] Daniel Garber (1992), p. 31. Garber offers a very thorough discussion of the *Rules*, pp. 31–50.

[4] CSM I, 7, 'Translator's Preface.'

The second set is concerned with problems that can be formulated in terms of equations, such problems being 'confined largely to the sphere of mathematics'.[5] The third set was supposed to deal with problems whose data is varied and complex, so much so that they cannot be formulated in terms of equations. Many physical, and all biological, problems fall into this category. The project was never completed.[6] I accept Murdoch's summary of the text.

The method of analysis expounded in the *Rules* requires that when we examine the ideas that come by way of the senses, we must first separate one idea out from another—we must make our ideas *distinct*.[7] Now, the *Rules* was written during a time when Descartes allowed ideas corporeal status (in addition to mental status).[8] Scholars usually try to sidestep this fact. But, the texts are there. For example, he says that the corporeal imagination is 'a genuine part of the body, and is large enough to allow different parts of it to take on many different *figures* and, generally, to retain them for some time'.[9] In Rule Fourteen, he reminds the reader of a discussion in Rule Twelve in which 'we conceived of the imagination, along with the ideas *existing in it*, as being nothing but a real body with real extension and shape'.[10] In the *Treatise on Man*, a book on which Descartes was working after abandoning the *Rules*, he tells us that those corporeal *figures* drawn on the pineal gland's surface are *ideas*:

> Now among these figures, it is not those imprinted on the external sense organs, or on the internal surface of the brain, which should be taken to be ideas—but only those which are traced in the spirits on the surface of the gland [that is, the pineal gland].[11]

So, in the *Rules* when he says that we must try to make our ideas distinct, it would not be unreasonable to take him as having had in mind, in addition to purely mental ideas, those figures located on the surface of the pineal gland.[12] Though not Descartes's mature view, assuming for sake of explanation that he included those corporeal figures among ideas, even getting a *corporeal* idea distinct seems completely at home within the context of the *Rules*—since the ideas on the corporeal imagination will in many cases overlap, resulting in an idea's being *confused* with another—literally, the ideas are *confusio*, or 'mixed together'.[13] We make an idea on the gland distinct, he says, if we succeed in separating it from any other ideas that may also be on the gland.[14] Thus, even in these early works, supposing that early in his career he allowed for the corporeality of ideas, *distinctness* (as it was also in the *Geometry*) is cast as the isolating or separating of ideas (in the *Geometry* it was lines), which is contrasted with *confusion*.

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[5] CSM I, 7.

[6] CSM I, 7. Also, see Garber, *Descartes' Metaphysical Physics*, p. 31. I am struck by the similarity of Descartes's project and what Aquinas says about the three sciences—metaphysics, mathematics, and physics. It seems to me that, with perhaps the exception of the first section, his sections of rules align with Aquinas' sciences. In particular, set two deals with mathematics, set three with physics. Since Aquinas seems to have used metaphysics and theology interchangeably, Descartes in the *Rules* stayed clear of metaphysics.

[7] AT X, 417–18; CSM I, 43.

[8] We find him toying with the view as late as the *Discourse*. The qualities of bodies, he says, 'can imprint various ideas on the brain' (AT VI, 55; CSM I, 139). For more on corporeal ideas in Descartes's work, see Edward Reed's 'Descartes' Corporeal Ideas Hypothesis and the Origin of Scientific Psychology', in *René Descartes: Critical Assessments*, 4 vols, George Moyal (ed.) (London: Routledge, 1991), vol. IV, pp. 244–62.

[9] AT X, 414; CSM, I 42.

[10] AT X, 441l; CSM, I 58, emphasis mine.

[11] AT XI, 176; CSM I, 106.

[12] For an argument for thinking that Descartes did not take the figures drawn on the surface of the gland to be ideas, see Dennis Sepper, 'Cartesian Imaginations: The Methods and Passions of Imagining', in *A Companion to Rationalism*, edited by Alan Nelson (Oxford: Blackwell Publishers, 2005), pp. 159–60.

I am only focusing temporarily on his remarks about corporeal ideas for heuristic reasons. I am in full agreement with scholars about his holding that ideas are modes of thinking substance, certainly with respect to his more mature view. But, starting with what he says about the concepts of clarity and distinctness in the context of discussions about those corporeal ideas and the pineal gland, I think, facilitates an explication of the concepts of clarity and distinctness as they evolved into his more mature view (where ideas are taken strictly to be modes of mind). So, I will drop the corporeal-idea talk once we have an account of clarity and distinctness that fits with the view that ideas are modes of thinking substance.

Let's return to the *Treatise On Man*. We are told that the corporeal imagination is located on the surface of the pineal gland. In making our ideas distinct, it appears that we must move the little gland about so that we isolate one pattern or figure (idea) on its surface from others that may be present. Now, given that the gland is surrounded on all sides by the nerve openings that constitute the interior surface of the brain, making our ideas that come by way of the senses distinct will be practically impossible.[15] For, there will be no place to move the gland so that one can prevent other ideas being drawn on the gland's surface.

Descartes's suggestion in the *Rules*—that in order to make our ideas clear and distinct, we need the ability to focus on one idea while ignoring the other ideas that are also present on the gland's surface—foreshadows the above suggestion made in the *Treatise On Man*.[16] By bringing an idea into focus, or by paying strict attention to it, we make the idea *clear*.[17] Paying attention to an idea for this purpose is possible only when we can 'see' the various relations that the idea has to other ideas (we need to see where it is drawn on the gland, for instance, relative to the other ideas). Once we make an idea clear, we can then go on to

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[13] AT X, 417–8; CSM I, 43.

[14] AT X, 417; CSM I, 43.

[15] For more details of this, see my *Dialogue* article, Smith (2001).

[16] '[I]n order to deduce a single thing from a collection of things—a frequent task—we must discard from the ideas of these things whatever does not demand our present attention, so that the remaining features can be retained more readily in the memory' (AT X, 417; CSM I, 43).

[17] I choose this term in light of what he says in the *Principles*: 'I call a perception "clear" when it is present and accessible to the attentive mind—just as we say that we see something clearly when it is present to the eye's gaze and stimulates it with a sufficient degree of strength and accessibility' (AT VIIIA, 22; CSM I, 207).

try to make it distinct (we can then begin the process of trying to isolate it from the others). But sometimes, especially in the cases of those ideas that come by way of the senses, we will not be able to make our ideas distinct. This is so, as was just noted, because it will be physically impossible to move the gland about in such a way so as to separate out the idea (the one that we are trying to make distinct) from the other ideas that are also present on the gland. It is possible, then, to make an idea clear, in that we pay strict attention to it (which amounts to our seeing the various relations that it has to other ideas on the gland), but fail to make it distinct, in that we cannot, by moving the gland, separate it out (or isolate it) from the other ideas that overlap it. However, if we have made an idea distinct, we will have also made it clear, since in order to separate it out from the other ideas, or, at least in order for us to *know* that we have done this, we must be (or must have been) paying strict attention to it (or certainly, we must now see where the distinct idea is relative to any others on the gland's surface).

According to Descartes, scientific knowledge requires that our ideas be clear and distinct. And so, given *only* the faculties of sensation and imagination (faculties involving the surface of the gland), we cannot for the physiological reasons just mentioned acquire scientific knowledge. To acquire such knowledge, we must be capable of going *beyond* our senses and imagination. The method developed in the *Rules* is intended to help us do just that. The method—which Descartes calls *analysis*—will guide us in our efforts to make our ideas (ideas now considered as purely mental modes) clear and distinct, without having to deal specifically with the ideas drawn on the surface of the gland.

Here, analysis does for Descartes what it was supposed to do for Plato. Recall (from Part I) that in his discussion of the divided line, Plato uses the method of analysis to move 'up' the line, so to speak, from objects qua originals (in the present context, they are objects *as bodies*), which are found in the Visible realm, to objects qua images, which are found in the Intelligible realm, eventually to the Forms (the first principles). In Cartesian terms, we might cast this as follows: we start by considering sensory experience, which involves us in a study of those ideas on the gland, and which results in our determining the constellation of ideas related to *the* idea of corporeal substance. This marks the introduction of the (purely mental) idea of body, and the simple natures that make it up. Descartes will later cast this pure idea of body as innate. This is the Platonic move from the Visible (or sensible) to the Intelligible. Analysis takes us further, however, showing us how the simple natures are ordered, and how this ordered system is rooted in first principles, which in his later work will be the innate ideas.

The particular method of sorting ideas that seems to have captured Descartes's attention, at least initially in his philosophical career, is found in the combinatorial work of the Spanish theologian, Raymond Lull. The *Ars Brevis*, which Descartes mentions in early letters to Beeckman, would also capture the attention of the young Leibniz, whose 1666 *Dissertatio de Arte Combinatoria* not only mentions Lull's method but criticizes certain aspects of it. (I deal more with Leibniz and Lull in Part III.) In a letter to Beeckman dated 29 April 1619, Descartes writes:

> Three days ago I had a conversation about Lull's *Ars Parva* with a learned man whom I met in an inn at Dordrecht. He boasted that he was able to apply Lull's method, and to do this so skillfully that he could talk for a whole hour on any subject you cared to mention; and if he was required to talk for a further hour on the same topic, he would find fresh things to say, and could even continue for twenty-four hours at a stretch.[18]

Descartes questions the man, seeing 'whether this method consists in arranging dialectical headings in a certain way and deriving arguments from them'.[19] The man replies that it does, but insists that there is *more* to the method than this. Though in his letter to Beeckman, Descartes admits that he does not have Lull's book, he seems to think that Beeckman does (or certainly thinks that it is something he could get his hands on), and in the letter asks him to look at the book to see whether the method can do what the man had claimed.

Lull's *Ars Brevis*, which Descartes refers to in the above passage as the *Ars Parva*,[20] purports to provide a universal method of solving problems.[21] Although there is no way of knowing whether Descartes ever read the book, he demonstrates that he was at least aware of its basic aim and method. For example, he states in an earlier letter to Beeckman, dated 26 March 1619:

> Let me be quite open with you about my project. What I want to produce is not something like Lull's *Ars Brevis*,[22] but rather a completely new science, which would provide a general solution of all possible equations involving any sort of quantity, whether continuous or discrete, each according to its nature.[23]

It is not clear which upcoming work of his he is referring to—the *Geometry* or the *Rules*—but whatever it was, in this earlier letter, Descartes assumes some acquaintance with Lull's method; enough to at least *contrast* it to the one he wants to develop. Given what he says in the later April letter, however, the conversation with that man at the Dordrecht inn got him to rethinking this earlier remark to Beeckman.

Lull was a scholastic logician of the thirteenth century, his name associated with others, such as Peter of Spain, Roger Bacon, and Aquinas's teacher, Albert the Great.[24] The works of these logicians were well known to Descartes and to his contemporaries. They (that is, Lull, Peter of Spain, Roger Bacon, and so on) were strong supporters of Aristotle's logical works. In the case of Lull, an emphasis was placed on the method of inquiry developed in the *Topics*.[25] But there is a twist here, for Lull appears to have been captivated by Eriugena's *On the Division of Nature* and Plato's *Timaeus*.[26] So, the method that he developed was not a typical Aristotelian system of terms and syllogisms. Rather, his system worked towards 'dividing' reality, resulting in a basic set of *elemental* concepts, which, through a mechanical wheel he developed to aid in the method, could be *combined* (by way of an algorithm that the operator of the wheel followed), resulting in the production of new propositions. It was nothing short of an early computational system.

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[18] AT X, 164; CSM III, 5.

[19] AT X, 164; CSM III, 5.

[20] According to Gaukroger, Descartes refers to Lull's *Ars Brevis* as the *Ars Parvis*. Both 'brevis' and 'parva' mean 'short'. So, the mistake is understandable. (Gaukroger, *Descartes: An Intellectual Biography*, 'Notes to pp. 100–108', p. 433, Note 94.) CSM suggests that the *Ars Brevis* is part of Lull's *Compendium On Method*, written in 1308 and printed in 1481 (CSM III, fn. 4). Gaukroger refers to this work as Lull's *Ars Inveniendi Veritatem*, 'The Art of Finding the Truth', which contained about 260 works. According to Gaukroger, the *Ars Brevis* was revived in a commentary written by Cornelius Agrippa, *De Incertitudine et Vanitate de Scientariusm et Artium*, published in 1527.

[21] CSM III, 2, fn.

[22] Here, Descartes correctly refers to Lull's work as the *Ars Brevis*.

[23] AT X, 156–7; CSM III, 2.

Lull was not the only famous logician known to Descartes who developed a method of conceptual division. The French logician, Peter Ramus, for instance, following those interested in a method of collection and division, developed a method of analysis and, like Lull, approached the inquiry of a subject by 'dividing' it into classes. In the preface of *Institutionum Logicorum Libri Duo* (1645), Ramus' method is cast as allowing its practitioner 'to define each art or curriculum subject with supreme assurance, to divide it into two "parts", redefine each part, subdivide the parts into two more "parts", and so on until he reached the atomic particles which admitted no further halving'.[27] We find this sort of approach described by Descartes in Rules Five, Six, and Seven of the *Rules*. About Rule Five, he says:

> The whole method consists entirely in the ordering and arranging of the objects on which we must concentrate our mind's eye if we are to discover some truth. We shall be following this method exactly if we first reduce complicated and obscure propositions step by step to simpler ones, and then, starting with the intuition of the simplest ones of all, try to ascend through the same steps to a knowledge of all the rest.[28]

About Rule Six, he says:

> Although the message of this Rule may not seem very novel, it contains nevertheless the main secret of my method; and there is no more useful Rule in this whole treatise. For it instructs us that all things can be arranged serially in various groups, not in so far as they can be referred to some ontological genus (such as the categories into which

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[24] Walter J. Ong, *Ramus: Method, and the Decay of Dialogue*,(Massachusetts: Harvard University Press, 1958), p. 56.

[25] See, for example, *Topics*, Book I, Chapter 1, 100a1–25.

[26] Frances Yates, *Lull & Bruno*, vol. 1 (London: Routledge and Kegan Paul, 1982).

[27] Ong, (1958), pp. 30–32.

[28] AT X, 379, CSM I, 20.

philosophers divide things), but in so far as some things can be known on the basis of others.[29]

And, about Rule Seven, he says:

> In order to make our knowledge complete, every single thing relating to our undertaking must be surveyed in a continuous and wholly uninterrupted sweep of thought, and be included in a sufficient and well-ordered enumeration.[30]

The division and subsequent arrangement of the various groups (which I shall show below are classes) related to a given investigation of objects is what Descartes calls an 'enumeration'.[31] We make the enumeration by beginning with the most complex things, reducing them by division into simpler parts. In Part I, the resulting schema was cast as an *hierarchical conceptual structure*—the reader may recall the structures (or schemes) of coloured marbles and geometrical figures. Even so, Descartes makes it clear that he is not interested in establishing categories based upon an ontological division, as we find, for instance, in Aristotle's *Categories*. Rather, he is interested in establishing categories based upon the order of knowledge, as we find in Aristotle's *Topics*, which may explain his initial interest in Lull's method. For example, if in order to know A we must first know B, then B is epistemologically (though perhaps not ontologically) prior to A. The enumeration should make this relation of priority explicit. The purpose of Rules Five to Eight is to show us how to establish for a given investigation the proper enumeration. Rules Nine to Twelve tell us how to apply the enumeration once we have established it.

Here is an example of how the method is supposed to be related to acquiring knowledge of a triangle. Descartes begins by saying:

> I can have knowledge of a triangle, even though it has never occurred to me that this knowledge involves knowledge also of the angle, the line, the number three, shape, extension, etc. But that does not preclude our saying that the nature of a triangle is composed of these other natures and that they are better known than the triangle, for it is just these natures that we understand to be present in it.[32]

The enumeration should *show* (structurally) that the angle, the line, the number three, shape, and extension are epistemologically prior to the triangle. Of course, we can have *some* knowledge of the triangle and still be ignorant of many things that constitute a fuller or more complete knowledge of it. A child, for instance, can be introduced to the triangle by way of the senses, and know that it is distinct, say, from a circle. In this sense, the child may be said to have, in a limited sense, knowledge of the triangle. But upon a closer inspection of his or her idea of the triangle, the child can discover that many other things constitute a deeper understanding of it. The child can come to know more about the triangle by coming to know more about angles, lines, numbers, and so on. But, that these things are *learned* later (temporally) in connection to his or her learning about triangles does not entail that they are epistemologically posterior. The child's fullest knowledge of the triangle will require knowledge of these other things. It is in the sense of being *required* that they are epistemologically prior.

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[29] AT X, 381; CSM I, 21.

[30] AT X, 387; CSM I, 25.

[31] AT X, 387–8; CSM I, 25.

[32] AT X, 422; CSM I, 46.

In order to establish an enumeration, we must first determine what is 'absolute' in the inquiry. Descartes says, 'I call "absolute" whatever has within it the pure and simple nature in question . . . I call this the simplest and the easiest thing when we can make use of it in solving problems.'[33] 'The "relative", on the other hand, is what shares the same nature, or at least something of the same nature, in virtue of which we can relate it to the absolute and deduce it from the absolute in a definite series of steps.'[34] And so, take geometrical figures, which are said to be the sorts of things they are in virtue of their shapes. In an inquiry into geometrical figures, the category (or concept) *shape* might be considered an absolute. We can then compare and contrast geometrical figures to one another, and categorize together those that share the *same* shape. Our inquiry will result in producing classes of geometrical figures—circles, triangles, squares, and so on. These classes are created by relating objects to one another, where the relation itself is understood in terms of what is absolute in the inquiry. 'The concept of the "relative" involves . . . [what] I call "relations".'[35] In the study of geometrical figures, the absolute is *shape*, and the relation that is employed in the inquiry is *x is the same shape as y*.

Although applying certain kinds of relations solves a problem with respect to sorting or enumerating the geometrical figures, a new problem arises. For, if the number of classes of geometrical shape approaches infinity, how is a study of them more helpful than a study of the particular shapes themselves? If Descartes is to be able to say anything certain about a large number of objects being investigated, he must be able to limit the number of classes to a finite, manageable number, certainly a number that is significantly smaller than the number of particular objects being investigated.[36] For, one of the benefits of producing an enumeration, he tells us, is that 'by means of enumeration nothing will wholly escape us and we shall be seen to have some knowledge on every question'.[37] About this feature, he says:

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[33] AT X, 381–2; CSM I, 21. The concept of the absolute is, I think, importantly connected to the concept of unity, or as Descartes sometimes calls it, *common nature*.

[34] AT X, 382; CSM I, 21.

[35] AT X, 382; CSM I, 21.

[36] In Rule Seven, Descartes offers the following as an example: 'say I wish to show by enumeration that the area of a circle is greater than the area of any other geometrical figure whose perimeter is the same length as the circle's. I need not review every geometrical figure. If I can demonstrate that this fact holds for some particular figures, I shall be entitled to conclude by induction that the same holds true in all the other cases as well' (AT X, 390; CSM I, 27). Here, he must have in mind an enumeration whose classes are circle and polygon. All geometrical figures can be categorized under these two classes. Hence, the inquiry, at this level (ignoring the infinite number of subsets of the class of polygon), is, according to Descartes, doable, since the only classes that need to be considered are two.

> In this context enumeration . . . consists in a thorough investigation of all the points relating to the problem at hand, an investigation which is so careful and accurate that we may conclude with manifest certainty that we have not inadvertently overlooked anything. So even though the object of our inquiry eludes us, provided we have made an enumeration we shall be wiser at least to the extent that we shall perceive with certainty that it could not possibly be discovered by any method known to us.[38]

Connected to this remark, he continues in Rule Seven:

> I said also that the enumeration must be well-ordered, partly because there is no more effective remedy for the defects I have just listed than a well-ordered scrutiny of all the relevant items, and partly because, if every single thing relevant to the question in hand were to be separately scrutinized, one lifetime would generally be insufficient for the task . . . But if we arrange all of the relevant items in the best order, so that for the most part they fall under definite classes, it will be sufficient if we look closely at one class, or at a member of each particular class, or at some classes rather than others. If we do that, we shall at any rate never pointlessly go over the same ground twice, and thanks to our well-devised order, we shall often manage to review quickly and effortlessly a large number of items which at first sight seemed formidably large.[39]

Though the example I am about to employ is not Descartes's, it will help to illustrate the benefit expressed in the above passages. Consider the argument:

1) If Smith is a human being, then he's mortal.
2) Smith is mortal.
3) Hence, he's a human being.

Suppose that the premises and conclusion of this argument are true. As we know, an *English argument*[40] is valid if, and only if, it has a valid argument form. Otherwise, an English argument is invalid. And, an *argument form* is valid if, and only if, there is no English argument that has this form such that its premises are true, and its conclusion is false. Otherwise, an argument form is invalid. The argument form of the above English argument is:

1) $P \rightarrow Q$
2) Q
3) So, P

Now, given that the premises and conclusion of the above English argument about Smith are true, it is not obvious whether the argument is valid. One way of determining whether the argument is valid or not is to see whether there is another English argument that has the same form as the argument about Smith, where the premises of this second argument are true, and the conclusion is false.[41] For example:

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[37] AT X, 387–8; CSM I, 25.

[38] AT X, 388–9; CSM I, 26.

[39] AT X, 391; CSM I, 27.

[40] Substitute for 'English' any natural language. I focus on English here simply because the example is taken from English.

1) If Zero (my cat) is a human being, then she's mortal.
2) Zero is mortal.
3) Hence, Zero is a human being.

Here, the premises are true and the conclusion is false. Since there is at least one English argument that takes this form, such that its premises are true and its conclusion is false, this argument form is invalid. Hence, *any* English argument that takes this form is invalid.[42] And so, the above argument about Smith, though its premises and conclusion are true, is invalid.

Now, consider a new argument:

1) If Smith is a human being, then he's mortal.
2) He's a human being.
3) Hence, he's mortal.

As we did just a moment ago, we can determine the invalidity of this argument by coming up with another English argument that has the same form as this one, where its premises are true and its conclusion is false. But try as we might, we do not come up with one. Does it follow from this that there is no such argument? No. Our not being able to immediately come up with one does not by itself rule out the possibility that there is not such an argument.

If we are clever, we can devise a way of evaluating arguments, such that we can determine without any doubts whether an argument is valid or invalid. For example, consider the above argument conceived strictly in terms of its argument form:

1) $P \rightarrow Q$
2) P
3) Q

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[41] Clearly, there are better ways of determining validity. I choose this method in order to make clearer the notion (and purpose) of an enumeration.

[42] It might be debated that the argument about Smith is valid, and that it takes an invalid form. But I am resisting this view, and contend that if a natural language argument takes an invalid form, even though this natural language argument has true premises and a true conclusion, then this natural language argument is invalid. I recognize that there are stronger objections to my view. For instance, suppose that an argument that took the form of Affirming the Consequent has a Tautology for a conclusion. Clearly, this argument cannot have true premises *and* a false conclusion, simply because its conclusion is a Tautology. Hence, it is valid. But I think that now it is a matter of what we take to be the determining factor for validity. I side with the logical form. I find it confusing to say that invalid forms of argument can have valid instances. But even if I adopt my objector's view of validity, the point that I shall make in what follows will not be affected.

Assuming that P and Q are statements, we can assign to them truth-values. In doing this, we can construct a truth-table:

Table 5.1

P	**Q**	**P→Q**	**P**	**Q**
T	**T**	**T**	**T**	**T**
T	**F**	**F**	**T**	**F**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**F**	**F**

Here, in our enumeration (or scheme) of all *possible* interpretive variations of the argument form, we see that no matter what the truth-values of P and Q are, there is no way of assigning truth and falsity to P and Q such that the premises of the argument are true and the conclusion false. In other words, although there is an infinite number of English arguments that take this form (or as Descartes says, 'even though the object of our inquiry eludes us' and is 'formidably large'), given the enumeration of possible interpretive variations above, which requires us to consider only four classes of argument, we can tell immediately that the argument form is valid, and hence, that any English argument that takes this form is valid. This method of evaluation has turned what appeared to be an endless task into a manageable, easy one.

Descartes's insight into the benefit of producing an enumeration is, I think, similar to our insight into the benefit of producing the truth-table. It can take what might be an indefinite number of objects—the number of *bodies* in the universe, for instance—and allow us to say something about each and every one of them without our having to look at each and every one of them. Now, contrast the way I am taking *enumeration* here to Curley's way of taking it (as simply list making). The truth-table, for example, is not a list of English arguments. Rather, it is a category schema of possible arguments of a certain form, a schema constructed from atomic elements and combinatorial rules. My view better aligns with how Descartes conceived knowledge (*scientia*), than Curley's. For, if by knowledge (*scientia*) Descartes means a *system* of interconnected propositions (in the *Rules* this will be a system of interconnected simple natures), which form a hierarchy, each proposition (or simple nature) ultimately connected to some first principle, then knowledge is not a list. The enumeration, as I am construing it here, is a schema that exhibits the elements of the system and how they are interconnected. This is the system that we are after when engaged in doing analysis. It is the system that analysis is supposed to yield.[43] And so, to construct the enumeration is an important part in our constructing our system of knowledge.

In the 1666 *Dissertatio de Arte Combinatoria*, Leibniz also notes the value of being able to cast an infinitely large number of objects in structural terms, which in this context is to group them and assign to each group a single name. Though unlike Descartes he appears to have taken the single name to refer to an ontological *whole* (instead of an epistemological class). For example, he says:

> [G]iven a plurality of beings all of which are understood to have something in common; then, since it is inconvenient or impossible to enumerate all of them every time, one name is thought of which takes the place of all the parts in our reasoning to make the expression shorter.[44]

Anticipating what he would later find in a study of the *Rules*, Leibniz says something that looks similar to Descartes's Rules Five to Seven:

> Furthermore, the whole itself (and thus number or totality) can be broken up into parts, smaller wholes as it were. This is the basis of complexions, provided you understand that there are common parts in the different smaller wholes themselves. For example, let the whole be ABC; then AB, BC, and AC will be smaller wholes, its parts. And the disposition of the smallest parts, or of the parts assumed to be smallest (that is, the unities) in relation to each other and to the whole can itself also be varied.[45]

One example of the value in enumerating[46] is a calculation Leibniz makes of the possible Aristotelian syllogistic forms, which seems to foreshadow the sort of investigative value attributed to the truth-table. Beginning with the four categorical propositions—A, E, I, O—Leibniz sets to combining them in groups of three, for example, AAA, EAE, AEE, and so on, in order to calculate the number of possible categorical syllogistic forms.[47] The *mood* of a categorical syllogism consists of the letter names of the propositions that make it up. Take, for example, the following syllogism:[48]

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[43] AT X, 388; CSM I, 25. Here, Descartes says that the enumeration is required for making knowledge complete.

[44] Gottfried Leibniz, *Dissertatio de Arte Combinatoria* (1666), translated by Loemker (1969), p. 73. I shall refer to this as 'Leibniz (1969)'. I also used a pdf copy of the original text (i.e., the 1666 text), which was accessible online at the time of writing at *gallica* (gallica.bnf.fr).

[45] Leibniz (1969), p. 77.

[46] In the above passage Leibniz uses 'enumerating' in its ordinary sense of numbering or listing. But, setting the various parts of ABC in order to the whole and to one another is not numbering the parts—it is ordering them according to some hierarchical structure. So, he is talking about what I claim is enumerating in its technical sense. He is just not calling it that.

[47] Leibniz (1969), pp. 21–2.

[48] I am borrowing here almost completely from Patrick Hurley's discussion of syllogistic logic, in *A Concise Introduction to Logic*, 7th edn. (New York: Wadsworth Publishing Company, 1999),

Major Premise	*No P are M*	E
Minor Premise	*Some M are S*	I
Conclusion	*Some S are not P*	O

The mood of this syllogism is EIO. Leibniz calculates the possible combinations of categorical propositions with respect to their constituting syllogistic forms (4 x 4 x 4). The number is 64 possible moods.

Now, the *major term* is the predicate of the conclusion; the *minor term* is the subject of the conclusion; the *middle term* connects the two. So, for the above, P is the major term, S is the minor term, and M is the middle term. The *figure* of a categorical syllogism is determined by the location of the two occurrences of the middle term in the premises. There are four possible figures.

Figure 1	Figure 2	Figure 3	Figure 4
M P	P M	M P	P M
S M	S M	M S	M S
S P	S P	S P	S P

So, the above categorical syllogism is EIO-4. That is, its mood is EIO and its figure is 4.

Leibniz takes the four figures and sixty-four moods and calculates that there are 256 possible forms of categorical syllogism.[49] Now, although there is an infinite number of natural language arguments that can take any one of these forms, an inspection of those arguments can be limited by our considering only 256 cases. What is more, not every syllogistic form produced by way of the combinatorial procedure will be valid. The trick will be to determine first which forms are valid (which guarantee the truth of the conclusion given the truth of the premises), and then look to see whether the natural language argument under consideration has a valid or invalid form.[50] To be sure, this is a bit of work, but it certainly beats the alternative—looking at an infinite number of natural language arguments. This insight into the value of the method of collection and division, it seems to me, is akin to Descartes's insight expressed in the *Rules*. And, to foreshadow what we will see in Part III, Leibniz's method begins with the

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pp. 240–80. The view of Leibniz on offer here is my own. (I only rely on Hurley for the example and on how to take *mood* and *figure*, *major*, *minor*, and *middle terms*, and so on.) So, if anyone finds a problem with what I say here about Leibniz, the problem is mine, not Hurley's.

[49] Leibniz (1969), p. 24. Here, however, what Leibniz actually does is halve the number of moods, so he considers thirty-two moods, and doubles the number of figures, so he considers eight figures. In another related calculation, he calculates 512 possible combinations (p. 18). So, he seems to be demonstrating the *flexibility* of the art of combining more than giving a definitive answer to a logical problem.

[50] Leibniz's procedure does not allow him to answer the question of whether a particular form is valid. What he seems to be emphasizing is a way to think about limiting the number of cases that need to be examined, here limiting the number of forms to 256.

atomic categories—categorical propositions—and *combines* them into groups of three, which is the syllogistic form. The combinatorial method he demonstrates is what I will show is synthesis.

Let's return to Descartes. Instead of propositions to which we applied the truth-table method, in the *Rules* the method is applied to ideas. In particular, the enumeration is intended to sort out the *elements* of our ideas. Descartes calls these elements 'simple natures'. There is a peculiar remark that he makes in Rule Seven about the concept of *simple*. He says:

> [W]e term 'simple' only those things which we know so clearly and distinctly that they cannot be divided by the mind into others which are more distinctly known.[51]

Although he may be saying that a simple is utterly unanalysable, there is another way to read him. This second way takes him to be saying that simplicity is relative. We might construe simplicity this way: A and B are simple with respect to C whenever (1) A and B are required for knowing C, and (2) an analysis of A and B will not produce a knowledge of C that is any clearer than knowing C in terms of A and B. Thus, even though lines, angles, and the number 3 can undergo further analysis, if such an analysis would not produce knowledge of the triangle that is any clearer than our knowing the triangle in terms of lines, angles, and the number 3, then these latter things are simple with respect to the triangle. The idea of relative simplicity is foreshadowed in Rule Six, where Descartes says that from one point of view, one thing may be taken to be more 'absolute' than something else, whereas from another point of view, that same thing may be taken to be more 'relative'.[52] More on this shortly.

Our ideas exhibit to us a world that is composed of simple natures. *Extension* and *shape* are examples of simple natures.[53] Colours, sounds, tastes, heat, and cold, and all the other 'sensible qualities', also appear to count as simple natures. Some natures are 'simpler' than others.[54] As I will show in Part IV, in line with Nelson's reading of Descartes, there may be simple natures that are simpl*est*. These will be the natures of God, mind, body, and on Nelson's view, union. So, there may be atomic categories in the Cartesian enumeration that represent absolute simples—natures that cannot undergo further conceptual analysis.

Now, as stated above, one of the aims of the *Rules* is to show how to construct a categorizational scheme or structure—an enumeration—that explicitly shows how the various classes of simple natures are related to one another, such that 'all these relations should be distinguished, and the interconnections between them, and their natural order, should be noted'.[55] This allows one to take in at a single glance how the simple natures in one part of the enumeration are linked, for instance, to 'the one that is absolute in the highest degree, by passing through all the intermediate ones'.[56] The enumeration should express an *ordering* of the

[51] AT X, 418; CSM I, 44.
[52] AT X, 382; CSM I, 22.
[53] AT X, 418–9; CSM I, 44.
[54] AT X, 382; CSM I 22.
[55] Ibid.
[56] Ibid.

simple natures, where epistemological priority is the form that the ordering will take.[57] That which is 'absolute in the highest degree' is, I think, a first principle, or perhaps better stated, an atomic category of the system in which all others are rooted.

An enumeration, as I understand it, is similar to a partition:

> We make a *partition* P of a set S, whenever we divide set S, such that every subset in P contains at least one member of S, each member of S falls into precisely one subset in P, and the union of the subsets in P will be the set S.

The formal definition of a partition is:[58]

> Let S be a non-empty set. A *partition* of S, denoted by P, is a subset of the power set of S, $p(S)$, such that (where C denotes a class):
>
> (i) If $C \in \mathrm{P}$, then $C \neq \emptyset$.
> (ii) If $C_1 \in \mathrm{P}$, $C_2 \in \mathrm{P}$, and $C_1 \neq C_2$, then $C_1 \cap C_2 = \emptyset$.
> (iii) $\bigcup_{C \in \mathrm{P}} C = S$.

Consider those monochromatic marbles introduced in Part I. We make a partition of this set of objects if, and only if, we *order* the marbles in such a way that (i) every sub-classification that is made contains at least one marble, (ii) each marble belongs to exactly one sub-classification, and (iii) the union of all of the sub-classifications will be the original set of marbles on the floor.

One way to order the marbles, as we did with those geometrical figures, is to see whether or not any relations hold between them. For example, apply the relation '*x* is the same colour as *y*'. An application of this relation will result in placing every marble in a jar (marbles of the same colour in the same jar), and no marble will belong to more than one jar (because, a marble does not have more than one colour). A union of these classes (the jars) will be all the marbles on the floor. Hence, applying this relation results in our making a partition of the marbles on the floor.

The relation that we applied is called an 'equivalence relation'. Equivalence relations have three properties: they are (i) reflexive, (ii) symmetric, and (iii) transitive. The formal definition of an equivalence relation is:

> Let r be a relation in S. r is an equivalence relation in S if and only if the following conditions hold in r:
>
> (i) For each $a \in S$, $<a,a> \in r$. (r is reflexive).
> (ii) If $<a,b> \in r$, then $<b,a> \in r$. (r is symmetric).
> (iii) If $<a,b> \in r$ and $<b,c> \in r$, then $<a,c> \in r$. (r is transitive),[59]

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[57] AT X, 390–1; CSM I, 26–7.

[58] The definition comes from *Elements Of Set Theory*, Peter Zehna and Robert Johnson, (Boston: Allyn and Bacon, 1962), p. 71. The power set of S is the set of all possible subsets of S.

[59] This comes from Zehna and Johnson (1962), p. 67. (see footnote above).

x is the same shape as y (which we used in the geometrical figure case) and *x is the same colour as y* meet the above three conditions. Looking at (i), any marble *a* we took from the floor is such that it is the same colour as itself. Thus, the relation *x is the same colour as y* is reflexive. Looking at (ii), any two marbles *a* and *b* we took from the floor are such that if *a* is the same colour as *b*, *b* is the same colour as *a*. Thus, the relation is symmetric. Looking at (iii), any three marbles *a*, *b*, and *c* we took from the floor are such that if *a* is the same colour as *b*, and *b* the same colour as *c*, then *a* is the same colour as *c*. Thus, the relation is transitive. So, *x is the same colour as y* is an equivalence relation.

By contrast, take the relation *x is the father of y*. Take any male *a*, it will not be the case that *a* is his own father. Thus, the relation is not reflexive. This alone is sufficient to show that *x is the father of y* is not an equivalence relation. But, to work the example out to its end, take two males *a* and *b*: if *a* is the father of *b*, *b* cannot be the father of *a*. Thus, the relation is not symmetric. Finally, take males *a*, *b*, and *c*; if *a* is the father of *b*, and *b* the father of *c*, it will not be the case that *a* is the father of *c* (*a* would be *c*'s grandfather).[60] Thus, the relation is not transitive. And so, as we said, the relation *x is the father of y* is not an equivalence relation.[61]

Why are equivalence relations significant here? As mathematicians would later prove, an equivalence relation in a non-empty set *S* will always determine a partition of *S*, and conversely.[62] And so, if we apply an equivalence relation to a set of objects that we wish to study, the application will produce a partition of those objects. The right sort of partition may be useful, since as the truth-table case showed, a partition may make manageable a study of an indefinitely large number of objects. But what is more, if the objects are partitioned (prior to investigation), there is in fact an equivalence relation we can discover that holds between the objects, a relation that if applied would produce (and reproduce) this particular partition.

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[60] I am ignoring the possibility that *c* is a clone of *b*.

[61] Descartes says that enumerations can be limited to the specific subject-matter one wishes to study (AT X, 390; CSM I, 26). He also recognizes that in some cases the enumeration will not be complete or distinct. As I understand his view, this remark is not intended to speak against the using of equivalence relations to do the sorting.

[62] For this, see Zehna and Johnson (1962), pp. 73–4.

6

The General Theory Continued

Recall the discussion of Plato from Part I. In his method of analysis he applied the sorting principle *like with like*, which, it seems to me, is an early insight into the connection between *equivalence relations* and the producing of *partitions*. Descartes also makes the connection. For example, in Rule Fourteen, he says:

> This common idea [which we recognize in different subjects or things] is carried over from one subject to the other solely by means of a simple comparison, which enables us to state that the thing we are seeking is in this or that respect similar to, or identical with, or equal to, some given thing. Accordingly, in all reasoning it is only by means of comparison that we attain an exact knowledge of the truth . . . [as a consequence] the syllogistic forms are [by themselves] of no help in grasping the truth of things. So it will be to the reader's advantage to reject them altogether and to think of all knowledge whatever—save knowledge obtained through simple and pure intuition of a single, solitary thing—as resulting from a comparison between two or more things . . . We should note that comparisons are said to be simple and straightforward only when the thing sought and the initial data participate equally in a certain nature.[1]

He adds to this that 'unity is the common nature, which, we said above, all the things which we are comparing must participate in equally'.[2] As in the *Geometry*, unity is cast as that which *unites* the things being investigated. In Rule Twelve, he introduces examples of what he calls *common notions*, which in this context express the properties of *common natures*, the common notions being those properties that account for a common nature's ability to sort objects into ordered classes. His examples of common notions are: 'Things that are the same as a third thing are the same as each other', and 'Things that cannot be related in the same way to a third thing are different in some respect.'[3] Although not perfect formulations, his examples of these sorts of properties (of relations) certainly flirt with the concepts of *reflexivity*, *symmetry*, and *transitivity*. It would not be too far fetched to suggest that Descartes's classifying objects according to their sharing a common nature is conceptually akin to Plato's *like with like* principle, where both appear to be ancestors of the insight into the producing of partitions by way of an application of equivalence relations.

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[1] AT X, 439–40; CSM I, 57. Here, I add the bracketed remarks to reflect what Descartes says in the larger context, in which we find this passage.

[2] AT X, 449; CSM I, 63.

[3] AT X, 419; CSM I, 45.

In *Der Logische Aufbau der Welt* (*The Logical Structure of the World*), Rudolf Carnap famously develops the idea of what he calls a 'constructional system', which:

> undertakes more than the division of concepts into various kinds and the investigation of the differences and mutual relations between these kinds. In addition, it attempts a step-by-step derivation or 'construction' of all concepts from certain fundamental concepts, so that a geneology of concepts results in which each one has its definite place. It is the main thesis of construction theory that all concepts can in this way be derived from a few fundamental concepts, and it is in this respect that it differs from most other ontologies.[4]

He historically locates the origins of such a system in Leibniz's ideas of a '*mathesis universalis* and of an *ars combinatoria*',[5] where 'the application of the theory of relations to the formulation of a constructional system,' he says, 'is closely related to Leibniz's idea of a *characteristica universalis* and of a *scientia generalis*'.[6] Carnap is right to identify Leibniz as an important player, but Descartes's part should not go unnoticed. As we know, the *Rules* was not published during Descartes's lifetime, but Claude Clerselier owned a copy of the manuscript, having come into possession of it presumably after Descartes's death in 1650. Leibniz purchased a copy of the original manuscript in Amsterdam in 1670.[7] In fact, the text on which Murdoch's English translation primarily relies is Leibniz's copy.[8] So, we can speculate that Leibniz was aware of Descartes's application of relations (common natures) to the making of a constructional system (enumeration) by at least 1670. Now, if Garber is right about when Descartes began to work on the *Rules*, Descartes's insight into such a constructional system came almost fifty years before Leibniz published his first book, the 1666 *Dissertatio de Arte Combinatoria*.

I mention the *Aufbau* here because it is considered by many to be an important contribution in the history of philosophy of science. The system that Carnap develops in the *Aufbau* is interestingly similar in some respects to the system Descartes attempted to develop in the *Rules*. As Alan Richardson characterizes the *Aufbau*:

> the task he [Carnap] sets for himself in the *Aufbau* is not the rational reconstruction of science, rather it is the rational reconstruction of epistemology. Carnap (§ 54) is attempting to order objects within a constructional system which captures their relative epistemic priority.[9]

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[4] Rudolf Carnap, *Der Logische Aufbau der Welt*, translated by Rolf A. George (Los Angeles: University of California Press, *translation* 1967), p. 5.

[5] Carnap (1967), p. 8.

[6] Ibid.

[7] AT X, 354. The history of the various copies floating around is given by Adam and Tannery, AT X, 351–7.

[8] CSM I, 'Translator's Preface', p. 7.

[9] Alan Richardson, 'How Not to Russell Carnap's Aufbau', PSA: Proceedings of the Bienial Meeting of the Philosophy of Science Association, vol. 1, p. 10.

Richardson's characterization of Carnap's *Aufbau* seems strikingly similar to Descartes's characterization of the *Rules*. In fact, Carnap's own characterization of his project, it seems to me, could easily have been something Descartes wrote about his own project in the *Rules*. Carnap writes:

> The present study is an attempt to apply the theory of relations to the task of analyzing reality. This is done in order to formulate the logical requirements which must be fulfilled by a constructional system of concepts, to bring into clearer focus the basis of the system, and to demonstrate by actually producing such a system (though part of it is only an outline) that it can be constructed on the indicated basis and within the indicated logical framework.[10]

And finally, the following sorts of similarities should not go unmentioned. For example, both see their respective systems as being 'generated' by comparing objects, using a special sort of relation—the equivalence relation—to determine the sorting of objects into classes.[11] Both see their respective systems as structurally hierarchical, the 'objects' (the term is being used broadly enough to denote classes) at one level in the hierarchy reducible to, and subsequently explained by, the 'objects' at another level (the latter being 'higher' in the hierarchy).[12] Both see their respective systems as ultimately 'grounded' by unanalysable simples. Both see their systems as yielding *structural* knowledge—that is, knowledge of x is simply knowledge of the elements a and b that compose x and the relation that a and b have to one another.[13] Finally, both see their respective systems as primarily logico-epistemological in scope, and not as systems of ontological (or metaphysical) classification.

Recall those marbles. When the marbles are separated from one another, in terms of their respective jars and using Descartes's terminology, they are *distinct*. Red marbles are distinct from blue marbles, and so on. With this in mind, and now thinking about *ideas*, a proper separation of the simple natures (analogues to those marbles) constitutes their being *distinct*. Descartes's connecting *separation* and *distinctness* will be carried over even to his later works—in the *Principles*, for example: 'I call a perception "distinct" if, as well as being clear, it is so sharply *separated* from all other perceptions that it contains within itself only what is clear.'[14] An enumeration *is* the structure that expresses the proper (epistemic) separation (and ordering) of the simple natures.

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[10] Carnap, *Aufbau*, Part I, Chapter A, §3, pp. 7–8.

[11] Carnap develops a much more complex system in this respect, for he sees important logical differences between classes, extensions, relational extensions, similarity circles, quality classes, as well as employing specific sorts of relations other than the equivalence relation, and so on. For Descartes on this, see also AT VI, 20; CSM I, 120–1.

[12] Descartes holds this even beyond the *Rules*. See, for example, the *Discourse*, AT VII, 18–19; CSM I, 120.

[13] Carnap discusses this, for example, in *Aufbau*, Part II, Chapter A, §11, p. 21. For Descartes, consider the discussion in Rule Twelve about explaining the difference between colours as differences in line-patterns and shapes (AT X, 413; CSM I, 40–1).

[14] AT VIIIA, 22; CSM I, 207–8, emphasis mine.

In the *Rules*, Descartes does not appear to discuss in any detail the notion of clarity. Earlier, we saw that Descartes says an idea is clear whenever we pay strict attention to it. We might enhance the little he does say in the *Rules* by looking to the *Principles*, for example, but this will be of no immediate help. For, there he says:

> I call a perception 'clear' when it is present and accessible to the attentive mind—just as we say that we see something clearly when it is present to the eye's gaze and stimulates it with a sufficient degree of strength and accessibility.[15]

I believe that there are places to look for help. His conversation with Burman is a place to begin. After a brief look at it, we can return to the *Rules* and to texts less controversial than the Burman text.

To Burman, Descartes contrasts clear *perception* with clear *imagination*. He says:

> Even though we can with the utmost clarity imagine the head of a lion joined to the body of a goat, or some such thing, it does not therefore follow that they exist, since we do not clearly perceive the link, so to speak, which joins the parts together.[16]

Unlike clear imagination, clear perception requires that we perceive the *connection* or *relation* between the elements of an idea. Now, this is interesting, since we know that Descartes discusses in the *Rules* the connection between comparing objects for similarity, which is understood as looking to see whether the objects being compared share a common nature. Burman presses Descartes further. He asks, 'But why should there not be a contradiction in the case of clear ideas which are inconsistent one with another? Take, for example, the combination of the idea of a finite being with that of an infinite being.'[17] Burman's question seems to be that if the idea of a finite being is clear, and the idea of an infinite being is clear, would not the complex idea of a finite-infinite being (which, I take it, would be akin to a square-circle) also be clear—even though no unifying relation *could* be perceived that would unify this contradictory nature? This, as I understand Burman's concern, would be a bad consequence of a theory of clarity. Descartes agrees, and answers:

> Even if those ideas are clear when taken apart, they are certainly not clear when joined together. Your idea is thus very obscure, for the conception you have of the combination and unity of the two ideas is not clear but extremely obscure.[18]

Given what Descartes says here, I want to suggest that an idea is *clear* whenever we perceive a relation that holds between the elements of the idea—a relation that *unifies* the idea's elements, where by *unifying* them I mean that it is shown how they stand to one another in the enumeration (simple natures that are related will either be members of the same class, or members of classes—*species*—that in turn are members of the same class—*genus*). Otherwise, the idea is obscure.[19]

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[15] AT VIIIA, 22; CSM I, 207.

[16] AT V, 160; CSM III, 343–4.

[17] *Descartes' Conversation With Burman*, translated by John Cottingham (Oxford: Clarendon Press, 1976), p. 25. I shall refer to this as 'Descartes (1976)'.

[18] Descartes (1976), p. 25.

We actually get examples of what I think are obscure ideas in the *Rules*. In Rule Twelve, Descartes writes:

> Deduction [construed as enumeration], therefore, remains as our sole means of compounding things in a way that enables us to be certain of their truth. Yet even with deduction there can be many drawbacks. If, say, we conclude that a given space full of air is empty, on the grounds that we do not perceive anything in it, either by sight, touch, or any other sense, then we are incorrectly conjoining the nature of a vacuum with the nature of this space . . . But it is within our power to avoid this error, *viz.* by never conjoining things unless we intuit that the conjunction of one with the other is wholly necessary, as we do for example when we deduce that nothing which lacks extension can have a shape, on the grounds that there is a necessary connection between shape and extension, and so on.[20]

So, relying on Descartes's assessment of the idea of a finite-infinite being in his discussion with Burman, I want to suggest that the ideas of an extended-nothing (a vacuum) and of a non-extended shaped thing, are not clear, because there is no relation to be found holding between the elements of the ideas. Thus, we see no relation that unites the elements into a single nature. These ideas are examples of obscure ideas.

We now know that A and B will be members of the same class in the enumeration if we can establish an equivalence relation holding between them—establishing this is to establish that they share a common nature. There appear to be at least two distinct forms of relatedness: *necessary connection* or *conjunction*, which Descartes mentions in the above passage, and *contingent connection* or *conjunction*. He says more about this in Rule Twelve:

> The conjunction is necessary when one of them [one of the *relata*] are implied (albeit confusedly) in the concept of the other so that we cannot conceive either of them distinctly if we judge them to be separate from each other. It is in this way that shape is conjoined with extension . . . because we cannot conceive of a shape which is completely lacking in extension . . .The union between such things [i.e., simple natures], however, is contingent when the relation conjoining them is not an inseparable one. This is the case when we say that a body is animate, that a man is dressed, etc.[21]

Although he does not say it here, I believe that the following is consistent with what he says. Now, let A and B be bodies. In comparing them in terms of their modes (properties), several classes will be formed. That is, in being finite bodies, both A and B will be shaped, will be sized, and so on. So, there will be the class of shaped things, the class of sized things, and so on, where shape and size are modes of body (extension). These modes, as we will see shortly, are established by measuring such things as A and B in certain ways. So, the schema here may look as follows: A and B will be members of the class of shaped things, as well as the class of sized things, and both classes—the classes of shaped and sized things—will be members (subclasses) of the larger class of *extended* things.

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[19] Not only must it be the case that a relation be perceived that unifies the elements of the idea, but some connection between this unified idea and other unified ideas must be perceived. I do not think that it is possible to have one, and only one, clear and distinct idea. Clarity and distinctness are systematic; they describe an interrelated scheme of ideas.

[20] AT X, 424–5; CSM I, 48.

[21] AT X, 421; CSM I, 45–6.

The relation between the class of shaped things and the class of extended things, if I understand Descartes correctly, is *necessary* insofar as the former implies the latter—that is, the class of shaped things is included in the class of extended things (we might express this: *all shaped things are extended things*). It is not clear to me whether the relation holding between the classes of shaped and sized things is contingent, and the examples Descartes gives us about a body's being animate or a man's wearing clothes is not obviously clear. But we can work up an example that will make this clearer. For instance, let A be a mind and B a body. As we will also see shortly, so conceived, A and B will be members of distinct classes in the enumeration (because they share no common nature). Since we will be able to conceive of the one class (the class of thinking things) independently of our conceiving of the other (the class of extended things), any relation we claim to hold between them will be contingent. Perhaps Descartes's examples, then, are to be understood thus: since we can conceive of body as not being animate, and of man as not being dressed, then being animate and being dressed are only contingently related to being body and to being man respectively.

Now, recall Descartes's above remark about relative simplicity. In this discussion he provides an example that is helpful and is relevant to the point I am trying to make here. Let me quote the entire passage now under discussion:

> The secret of this technique consists entirely in our attentively noting in all things that which is absolute in the highest degree. For some things are more absolute than others from one point of view, yet more relative from a different point of view. For example, the universal is more absolute than the particular, in virtue of its having a simpler nature, but it can also be said to be more relative than the particular in that it depends upon particulars for its existence, etc. Again, certain things sometimes are really more absolute than others, yet not most absolute of all. Thus a species is something absolute with respect to particulars, but with respect to the genus it is relative; and where measureable items are concerned, extension is something absolute, but among the varieties of extension length is something absolute, etc.[22]

So, now applying this to the above example in which A and B are bodies, the class of shaped things is absolute with respect to the particulars A and B, but relative with respect to the class of extended things, the class of shaped things being a member of the class of extended things. As I suggested a moment ago, what Descartes seems to be describing here is what I called in Part I *the hierarchical structure of the conceptual landscape*.

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[22] AT X, 38–83; CSM I, 22.

Establishing that A and B share a common nature is to establish an equivalence relation that holds between them. And, as we now know, to establish this is to establish that they are members of a common class. Now, I take it that by moving from thinking of A and B as individuals to thinking of them in terms of the class (of which both are members), we have in some sense *consolidated* what we have to think about. Whereas before applying the equivalence relation we were thinking of two things (here A and B), we are, after applying the equivalence relation, now thinking of only one (the class of shaped things). Descartes expresses another instance of the value of implementing this sort of consolidation move. In Rule Eighteen, for example, he says: 'Things are much clearer when they are brought under one single general precept rather than split up among many particular ones.'[23] Finding a certain kind of relation that holds between elements of an idea is a way of making the idea (its content) simpler, and hence clearer.[24] I will pick up this insight again in a moment.

As is well known, in the *Rules* Descartes introduces the concepts of intuition and deduction. Intuition is a certain kind of cognition, an 'indubitable' thought 'of a clear and attentive mind which proceeds solely from the light of reason'.[25] Deduction, on the other hand, is understood to be an inference that is made when something is believed to follow 'necessarily from some other proposition'.[26] In contrast to intuition, which is a single, simple cognition, deduction is 'a continuous and uninterrupted movement of thought in which each individual proposition is clearly intuited'.[27] He concludes, 'Hence we are distinguishing mental intuition from certain deduction on the grounds that we are aware of a movement of a sort of sequence in the latter but not in the former.'[28] The fact that the mind must move as it progresses from one proposition to another (as it does in following a proof from premises to conclusion) makes deduction somewhat less reliable than intuition, for there is always the possibility of making a *wrong* move.

Constructing the enumeration, I think, allows Descartes a way of turning deductions into a special sort of intuition—or perhaps better, into a cognition that is intuition-like. For, the enumeration somehow crystallizes the information being considered into a single, unified cognition, in the way A and B are consolidated into the single class of shaped things when seeing the common nature that they share. But what is more, we learn of the special 'necessary conjunction' relating the class of shaped things and the class of extended things, where the former is understood to be included in the latter. Consider this in light of what Descartes says further in Rule Twelve:

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[23] AT X, 461; CSM I, 71.

[24] This view finds expression later on. For instance, he tells Regius (in a 1640 letter) that the ability to perceive a universal is attributed 'to the intellect alone, which relates to many things an idea which is in itself singular' (AT III, 66; CSMK III, 148).

[25] AT X, 368; CSM I, 14. [26] AT X, 369; CSM I, 15. [27] Ibid.

[28] AT X, 370; CSM I, 15. Here, Descartes goes on to draw a further distinction, namely, that for deduction self-awareness is not required, but for intuition it is. I will ignore this difference here § though I believe that it is an important difference.

> First, we have explained distinctly and, I think, by an adequate enumeration, what at the outset we were able to present only in a confused and rough-and-ready way, *viz.* that there are no paths to certain knowledge of the truth accessible to men save manifest intuition and necessary deduction.[29]

Here, *necessary deduction* is no doubt the deduction one makes in 'seeing', for instance, that 'A is extended' necessarily follows from 'A is shaped', since one 'sees' that A is a member of the class of shaped things, and that the class of shaped things is a member of the class of extended things (the reader should be reminded of the discussion in Part I in which Aristotle's system of class inclusion yielded an anatomy of logical inference). The move is not horizontal, so to speak, across a field of distinct things. To be sure, in moving from thinking about A to thinking about B, or in moving from thinking 'A is shaped' to thinking 'A is extended', we appear to be moving from one thing to some other distinct thing. Rather, as I put it in Part I, the move is vertical, so to speak, which in this context is to be understood as our 'seeing' that A and B are included in a single class (e.g., the class of shaped things), and that the one concept (the class of shaped things) is included in the other (the class of extended things).

He continues in Rule Twelve:

> It is clear that mental intuition extends to all these simple natures and to our knowledge of the necessary connections between them, and in short to everything else which the intellect finds to be present exactly within itself or in the corporeal imagination.[30]

So, intuition can be extended to cover knowledge of the simple natures *and* their necessary connections. Intuition need not be of a single, unanalysable simple nature. This, I think, is related to a remark he makes earlier in Rule Eleven. There, he says:

> This is a good time to explain more clearly what was said about mental intuition in Rules Three and Seven. In one passage we contrasted it with deduction, and in another only with enumeration, which we defined as an inference drawn from many disconnected facts. But in the same passage we said that a simple deduction of one fact from another is performed by means of intuition.[31]

What I will suggest is that we read Descartes here as telling us that, although deduction is a 'movement' of the mind that can be contrasted to intuition (noted in Rule Three), enumeration is something other than deduction in this sense (noted in Rule Seven). Enumeration, I think, will turn out to be the sort of deduction that he says is performed by means of intuition. Seeing how things 'hang together'—that is, how the simple natures are interrelated (necessarily conjoined), is a kind of intuition, or is a cognition that is intuition-like.

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[29] AT X, 425; CSM I, 48.

[30] AT X, 425; CSM I, 48.

[31] AT X, 407; CSM I, 37.

Descartes continues:

> It was necessary to proceed in this way, because two things are required for mental intuition: first, the proposition intuited must be clear and distinct; second the whole proposition must be understood all at once, and not bit by bit.[32]

So, to clearly and distinctly perceive some truth is importantly part of what it is to intuit. And, to intuit something, this something must be understood all at once, not bit by bit, as he says. Thus, when clearly and distinctly perceiving the nature of the triangle, this view suggests that the presentation of the triangle's nature in the intellect must be simple. But, we know that the knowledge of the triangle requires knowledge of such things as lines, angles, and the number 3. What the enumeration makes possible, I think, is a unified conceptual picture of lines, angles, and numbers, which shows the necessary connections that hold between these elements—that is, the relations that unify them into a single nature—and so the nature of the triangle is presented in this sense as a single, unified whole. In light of difficulties that might arise in calling this a unified *whole*, or, as I called it a moment ago, a *picture*, we could think of it this way: the triangle's nature is presented to the mind as a *system* of interrelated simple natures.

Descartes continues in Rule Eleven:

> But when we think of the process of deduction as we did in Rule Three [i.e., deduction in contrast to intuition], it does not seem to take place all at once: inferring one thing from another involves a kind of movement of our mind . . . But if we look at the process as a completed process, as we did in Rule Seven [i.e., as enumeration], then it no longer signifies a movement but rather the completion of a movement.[33]

It is unclear what Descartes means exactly by a *completion* of a movement. I will suggest, as I did above, that one way to read this is to take him as saying that the enumeration presents to the mind a single, unified *picture*, of the system of simple natures, or in being more modest, a unified picture of how the simple natures under investigation hang together.[34] In calling it the *completion* of a movement, Descartes is no doubt suggesting that the intellect *no longer moves*. In this way, it is intuition-like.

I argued in an earlier paper, considering the possibility of ideas constituted of a single simple nature, that perceiving a relation holding *between* elements looks problematic for the sort of view on offer here, for if there is only one element, there would appear to be no relation to perceive.[35] And so, without some modification to the definition, such ideas could not be said to be *clear*. I there suggested two senses of clarity:

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[32] AT X, 407; CSM I, 37.

[33] AT X, 407–8; CSM I, 37.

[34] No doubt this is connected to how he wanted to take the *cogito* later in the *Meditations.* I have in mind what he says in the Second Replies, AT VII, 140; CSM I, 100.

[35] K. Smith (2001), pp. 290–91.

> The idea of A is *clear* only if A is an unanalyzable simple (the idea exhibits the element constituting the nature of A),

and,

> The idea of A is *clear* only if the idea exhibits the elements constituting the nature of A and it exhibits the relation unifying those elements.[36]

It seems to me, now, having reflected on this since the publishing of that paper, that even in the case of an idea constituted of a single simple nature, the single nature could certainly meet the conditions holding for the application of an equivalence relation. That is, if the idea exhibited the single nature *a* and some relation *R* (the relation of identity, for example) the conditions of reflexivity, symmetry, and transitivity would still be met. Moreover, in that paper I also worked out the sense in which clarity is a scalar concept—an account of how ideas can be more or less clear than other ideas. For my purpose here, it is neither necessary nor helpful to work out this feature. I refer the reader to the previous study on this topic if he or she is interested in seeing how the scalar feature gets worked out.[37] With that said, the above senses of clarity will be adopted for the remainder of the book.

It should not go unnoticed that there is a similarity between the above account of clarity and the one we found earlier in the *Geometry*, where an equation was said to be clear whenever it explicitly showed the relation or relations shared between the equation's parts.

In establishing a relation (a common nature) that holds between the marbles—a relation that unites them into classes—we are *clear*, but what is more, in establishing a relation that results in a genuine partition of the marbles, we are *clear* and *distinct*. This, I think, is what Descartes is getting at in his discussion of ideas. In establishing a relation that holds between the elements of our ideas—the simple natures—we are clear, and in establishing a relation that produces a partition or enumeration of those simple natures, we are clear and distinct. On my reading, then, our clearly and distinctly perceiving ideas is just another way of saying that our ideas are clear and distinct, where the latter features are shown by a certain ordering of the simple natures—the elements constituting the objects of our ideas.

So, let's put this all together. In the *Rules*, recall, Descartes tells us of the possibility of constructing an enumeration that explicitly shows how the various classes of simple natures are ordered.[38] Here is how I think that Cartesian enumeration is supposed to go. First, Descartes establishes the objects of investigation. They are the finite substances of the cosmos. For the sake of simplicity, call this set of finite substances *S*. Descartes first divides them according to what he would later call their 'principal attributes', *thinking* and *extension*.[39] The procedure goes as follows. Descartes applies the equivalence relation *x has the same principal attribute as y* to the objects of *S*, which yields a new set, call this *P*, whose two members are the *set of thinking things* (call this *T*), and *the set of extended things* (call this *E*). Since every member of *S* is in exactly one member of *P* (that is, every member of *S* is either a member of *T* or *E* but not both), and the union of the sets in *P* (that is, the union of *T* and *E*) is *S*, *P* is a partition of *S*. I will call this the *master enumeration.*

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[36] K. Smith (2001), p. 291.

[37] Ibid.

[38] AT X, 390–1; CSM I, 26–7.

There are attributes other than the principal ones that we could have considered. For example, there is *existence*, *unity*, and *duration*.[40] Since these are shared by *all* finite substances, and subsequently by *both* thinking and extended substances, understanding the objects of *S* solely in terms of these other attributes would not yield what I am calling the master enumeration—the enumeration dividing finite substances up into the classes of *thinking* and *extended* things.

Descartes warns us in Rule Seven that if we fail to properly distinguish from one another the objects under investigation, that is, if we fail to see them in light of the enumeration, our knowledge of them will be *confused*.[41] In line with this warning, and having now introduced the master enumeration, I want to update the earlier definition of *distinctness* and offer the following:

> An idea is *distinct* just in case its element is, or elements are, found in *E* or in *T*, but not in both.

Otherwise, an idea is confused (*confusio*)—literally, as stated before, its elements are 'mixed together'.

On this analysis, then, the innate idea of body is clear and distinct, since there is a relation that is perceived holding between its elements (for example, the elements denoted by 'size' and 'shape'), where these elements are enumerations (or, are members of enumerations) of *E*. They share the common nature *extension*—or perhaps more accurately, both satisfy *x modifies the same attribute as y*. (There are also several mathematical relations that we understand to hold between *size* and *shape*—expressed by the concepts *area* and *volume*; for example, which are also enumerations of *E*.) The same sort of analysis of our idea of the sun that comes by way of the senses would show that this idea is obscure and confused. The elements of the idea, denoted by 'circular' and 'yellow' to name two, cannot be intelligibly related to one another.[42] Since there is no relation that will unite these elements, the idea that includes them is obscure. Further, the elements belong to different classes of the enumeration. The element denoted by 'circular' is an enumeration (or member of an enumeration) of E, whereas the element denoted by 'yellow' is an enumeration (or member of an enumeration) of T.[43] Since the elements of the idea are found in *both* T and E, the idea is confused.

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[39] AT VIIIA, 25; CSM I, 210.

[40] AT X, 419; CSM I, 44–5.

[41] AT X, 390; CSM I, 26.

[42] By 'unintelligible' I mean that these elements do not logically entail one another, nor do they entail the same attribute. Circular entails extension, and yellow entails thought. See what Descartes says in *Principles*, Part IV, Article 198 (AT VIIIA, 321–3; CSM I, 284–5).

If what I have said is right, the method developed in the *Rules* for establishing the enumeration is a method for producing clear and distinct perception. Critics of this particular take on Descartes have argued that, although enumerating as laid out above *can* be found in the earlier *Rules*, it cannot be found in his more mature work, such as the *Meditations*. Thus, if his more mature work best reflects his philosophical view, enumerating is not something to focus on when working out Descartes's philosophical view. My reply to this is that enumerating does, in fact, play a vital role in the more mature work, and is found in the *Meditations*. I want now to turn to a close but brief look at the *Meditations* and locate enumerating, showing where Descartes employs it as the method of analysis (his method of getting clear and distinct), as I have laid it out here.

Now that we know what we are looking for, it should be rather easy to see where Descartes employs in the *Meditations* a variety of enumerations—which work towards his establishing what I shall call the master enumeration. Now, for a start, he uses enumerations in the First Meditation set-up of the method of doubt, in the Third Meditation proof for the existence of God, and in the Sixth Meditation proof for the existence of body. Let me briefly say something about each.

In developing the method of doubt, Descartes says:

> it will not be necessary for me to show that all my opinions are false, which is something I could perhaps never manage. Reason now leads me to think that I should hold back my assent from opinions which are not completely certain and indubitable just as carefully as I do from those which are patently false. So, for the purpose of rejecting all my opinions, it will be enough if I find in each of them at least some reason for doubt. And to do this I will not need to run through them all individually, which would be an endless task. Once the foundations of a building are undermined, anything built on them collapses of its own accord; so I will go straight for the basic principles on which all my former beliefs rested.[44]

The enumeration of his beliefs (in terms of their origins) is set against his recognizing that to work through each and every belief would be 'an endless task'—a concern echoed in Rule Seven of the *Rules*. He goes on to divide all of his beliefs into two classes: those acquired from the senses and those not acquired from the senses. (The equivalence relation here is *x has the same origin as y*, which is used again in the enumerating of his ideas in the Third Meditation.) Sense-doubt and dream-doubt call into question the former class of beliefs. Demon-doubt and hyperbolic-doubt call into question the latter class of beliefs.[45] By the end of the First Meditation, it *appears* that no belief is incapable of being doubted, though at the beginning of the Second Meditation we discover that there is at least one belief that cannot be doubted, which in turn leads to showing that there is an entire class of belief that cannot be doubted.

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[43] AT VIIIA, 35–6; CSM I, 219.

[44] AT VII, 18; CSM II, 17.

A second enumeration comes into play when, in the Third Meditation, Descartes divides his *thoughts* into kinds,[46] and this enumeration is expanded when he divides his *ideas*, a kind of thought, into kinds.[47] Descartes introduces yet another enumeration when he divides his ideas according to the degrees of objective reality they contain. As we know, the cause of the objective reality contained in an idea must have a level of formal reality that is greater than, or equal to, the level of objective reality contained in the idea. So, one enumeration may be importantly related to another. These enumerations usher in the Third Meditation proof for God's existence:

> If the objective reality of any of my ideas turns out to be so great that I am sure the same reality does not reside in me, either formally or eminently, and hence that I myself cannot be its cause, it will necessarily follow that I am not alone in the world, but that some other thing which is the cause of this idea also exists. But if no such idea is to be found in me, I shall have no argument to convince me of the existence of anything apart from myself. For despite a most careful and comprehensive survey, this is the only argument I have so far been able to find.[48]

The idea of God has an infinite level of objective reality. Since Descartes is a finite substance, his level of formal reality is not great enough to cause the level of objective reality contained in the idea of God. It necessarily follows, he says, that there exists something other than himself that is the cause of his idea of God. Notice that Descartes is banking on the idea that something is being classified in one enumeration (the level of objective reality in an idea, in this case) is necessarily connected to how something else is classified in another enumeration (the level of *formal reality* of the cause [of the objective reality in the idea]). Curley's miscasting enumerating as list making, considered earlier, might explain why he thinks that enumerating fails to have any pay-off with respect to this proof for God's existence. But, as I have shown here, enumerating plays a vital role in the proof.

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[45] Hyperbolic doubt is associated with his nature. '[Suppose that some others are right and] that I have arrived at my present state by fate or chance or a continuous chain of events, or by some other means; yet since deception and error seem to be imperfections, the less powerful they [these others] make my original cause, the more likely it is that I am so imperfect as to be deceived all the time' (AT VII, 21; CSM II, 14). Hence, I may believe that 2+2=4, and that it could not be otherwise. But, it may turn out that my nature is so imperfect that what I believe to be (necessarily) true, is in fact false. The imperfection of my nature will prohibit me from ever being able to determine whether such beliefs are *really* (necessarily) true.

[46] He divides his thoughts into kinds: ideas, volitions or emotions, and judgements (AT VII, 36–7; CSM II, 25–6) This enumeration allows him to determine which thoughts are the bearers of truth and falsity (AT VII, 37; CSM II, 25).

[47] He divides his ideas into kinds: innate, adventitious, and factitious (AT VII, 37–8; CSM II, 26). This enumeration eventually allows him to isolate the idea of God from his ideas that come by way of the senses, and from those that are invented by him.

[48] AT VII, 42; CSM II, 29.

In the Sixth Meditation, Descartes argues for the existence of body. He does this by considering the idea of body and an enumeration of all of the possible causes of the idea. The possible causes are himself, body, God, or some creature more noble than body (an angel).[49] By process of elimination, Descartes shows that neither himself, God, nor some creature more noble than body, can be the cause of his idea of body, and so by default the only possible cause is body. He concludes that 'corporeal things exist'.[50] As in the other cases (and much like the truth-table example), the conclusion rests on the fact that *every* possibility relevant to the inquiry has been considered. For if there is something that has not been considered, the elimination argument cannot work. The proper enumeration, then, is vital to the success of Descartes's elimination arguments, the Sixth Meditation proof for the existence of body being one of them.

It should be clear, then, that Descartes employs in the *Meditations* a method of analysis that is very similar to the one he had earlier developed in the *Rules*—a method of collection and division. Although perhaps driven by the dialectical style of narrative in the *Meditations*, the method of analysis as characterized here shows that it and the style of narrative are not one and the same. Even so, this part would not be complete if it could not be further shown how this method is connected to the concepts of *quantity* and *measurement*. This is, after all, the point of bringing the method to light. And so, I want to now turn to looking more carefully at the connections between the method of analysis and the concepts of quantity and measure, both of which are concepts that Descartes says are the exclusive concern of mathematics.

[49] AT VII, 79; CSM II, 55.

[50] AT VII, 80; CSM II, 55.

7

Enumeration, Quantity, and Measurement

Having established the master enumeration, Descartes appears to take each member of P and produces new enumerations of each in terms of how each can be 'modified'. For example, *operations of the intellect* (I) and *operations of the will* (W) are general categories of modifications of *thinking*. Thus, I and W are enumerations of T. These enumerations in turn can be enumerated further.[1] 'Dimensions', as Descartes calls them, are enumerations of modifications of *extension*. He says:

> By 'dimension' we mean simply a mode or aspect in respect of which some subject is considered to be measurable. Thus, length, breadth and depth are not the only dimensions of body: weight too is a dimension—the dimension in terms of which objects are weighed. Speed is a dimension—the dimension of motion; and there are countless other instances of this sort.[2]

And so, *size* (Z), *weight* (M), and *speed* (V), are dimensions, and are enumerations of E. Note that the dimensions (or modes) of extension are determined by *measuring*, and are not determined *a priori* by the armchair-philosopher, for 'whether dimensions have a real basis,' Descartes goes on to say, 'is something for the *physicist* to consider.'[3]

The enumeration of dimensions (that is, the enumerations of E) will be of ultimate interest to us in our coming to understand the 'mathematization' of early modern physics. But, before getting to this, I want to add one more piece to the puzzle. Let me start by noting that, even though Descartes tells us that the determining of dimensions is an empirical matter, there is a *logical* or *structural* relation holding between the *enumerations* of E (the dimensions of *size*, *shape*, *weight*, and so on), and E (the class of extended things)—for instance, size *entails* extension, shape *entails* extension, motion *entails* extension, and so on.[4] This is not an accident, of course. Rather, what the physicist has discovered in the investigation of bodies are equivalence relations that if applied *would* yield the particular partition of corporeal modes. Let me be clearer here before moving on.

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[1] AT VIIIA 17; CSM I 204.

[2] AT X, 447–8; CSM I, 62. In dividing the large subclass of extended things into dimensions, Descartes is able to ground his theory of equations. This would be developed more fully in the *Geometry*. In particular, Descartes says that dimensions and qualities are expressed by way of exponential notation. The exponent tells us how many dimensions are in play (AT VI, 371–3). I will look carefully at this in the next chapter.

[3] AT X, 448; CSM I, 63, emphasis mine.

[4] AT VIIIA, 25; CSM I, 210.

Start with the set of extended things E:

$$E = \{a, b, c, \ldots, x\},$$

where a, b, c, and so on, are 'bodies' whose principal attribute is extension—in other words, each is an example of a finite body. An enumeration of a *mode* of extension would be a collection and division of the members of E by recasting them in terms of the modification. So, let's start with size (Z), which is determined by some system of measure, and let's say that in looking at the sizes of the members of E we get the following partition:

$$\begin{array}{lccccc} & 1 & 2 & 3 & ,\ldots, & N \\ Z = \{ & \{f, g, k\}, & \{a, j\}, & \{b, c, r, o\}, & \ldots, & \{x, y, x\}\}. \end{array}$$

The relation we use is *x is the same size as y*, where, for instance, objects having 1 unit of size belong to a class (whose number above the class designates the quantity of size measured), objects having 2 units of size belong to a class, and so on. Thus, f and g have 1 unit of size, a and j have 2 units of size, etc. Of course, these classes would have more members than I show here. I am simply providing the above for the purpose of explanation. Since Descartes will reject the idea that a finite body could have an infinite size (it is by definition *finite*), in principle the sizes of objects will come to an end, which I have designated as N. So, x and y are instances of the largest objects in the cosmos.

Now, let's recast the members of E in terms of another mode—speed (V), for instance. So, we measure the objects according to this particular system of measure and produce the following partition:

$$\begin{array}{lccccc} & 1 & 2 & 3 & ,\ldots, & N \\ V = \{ & \{r, h, k\}, & \{n, i, w, q\}, & \{a, m, v, s\}, & \ldots, & \{l, c\}\} \end{array}$$

The relation we use to produce this partition is *x has the same speed as y*. Thus, the measured speed of n is 2 units of speed, the measured speed of a is 3 units of speed, and so on. Objects l and c, for instance, are among the fastest moving objects in the cosmos. Notice that V enumerates E differently in this case than Z enumerates E. Since a has 2 units of size, in Z we find it in the class designated by 2; since a has 3 units of speed, in V we find it in the class designated by 3. Each dimension, or mode of extension determined by some system of measure, will enumerate or partition E. The point to stress here is that Z and V are different ways of regarding members of E. And so, size is one way of regarding finite bodies, speed another way, weight another way, and so on. 'It is clear from this,' Descartes says, 'that there can be countless different dimensions within the same subject, that these add absolutely nothing to the things which possess them, and that they are understood in the same way whether they have a real basis in the objects themselves or are arbitrary inventions of our mind.'[5] That a dimension *adds* nothing to a body, I think, is Descartes's way of saying that it is a way of *regarding* a body. The 'regarding' aspect, however, does not entail anything like 'purely conceptual'. I will return to this claim in Part IV. For the moment, let's take Descartes at his word and entertain the claim that certain dimensions are *real*. 'The weight of a body is something real; so too is the speed of a motion.'[6] Their *reality* is determined by the fact that they are measurable—specifically, by ratio scales (which I discuss below). They are genuine modes of body.

Think back to that theorem which states that every equivalence relation will partition a set of objects, and conversely. The reality is *already* partitioned (by divine institution), and so what the physicist is discovering in determining which dimensions are modes of body are the equivalence relations that determine the partition. Could Descartes be thinking that they are the keys to unlocking the door that stands between the creature and the Creator, revealing to us the system of divine exemplars, the (conceptual) partition of the Divine Being? As Descartes puts it: 'For if nature is considered in its general aspect, then I understand by the term nothing other than God himself, or the ordered system of created things established by God.'[7]

Recall that, in the *Principles*, Descartes said that the quantity or measure of *ten feet* is 'unintelligible without some extended substance of which it is the quantity'.[8] Connected to this remark, David Lachterman says:

> As we saw in the details of Descartes' geometry, not all algebraic equations can be constructed or represented phenomenally. Those that can do not require any particular external body or discrete bodily shape to validate them; they do require, however, a receptacle or a phenomenal continuum, not of the mind's making, in which successful constructions can make their appearance.[9]

In Part III, I will discuss in more detail Lachterman's *phenomenal continuum* remark by way of Vincenzo De Risi's analysis of Leibniz. At present, I want to focus on Lachterman's use of *receptacle*, a term that the reader will recall, from Part I, was introduced by Plato in the *Timaeus*. For Plato, the receptacle was taken to be extension in three dimensions. In Part IV, I argue that the existence of matter (Cartesian extension) is a necessary condition for the intelligibility of mathematics, which, I believe, is compatible with Lachterman's asserting, as he does in the above passage, that the receptacle (i.e., Cartesian extension) cannot be simply 'of the mind's making'. So, more on these things later.

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[5] AT X, 448; CSM I, 62–3.

[6] Ibid.

[7] AT VII 80; CSM II 56.

[8] AT VIIIA 45; CSM I 226.

[9] David R. Lachterman. '*Objectum Purae Matheseos*: Mathematical Construction and the Passage from Essence to Existence', in *Essays on Descartes'* Meditations, edited by Amélie Oksenberg Rorty (Los Angeles: University of California Press, 1986), p. 453.

As Descartes says in the *Principles* passage just quoted, *some* extended substance is required for the intelligibility of quantity and measurement. This is not the only text in which Descartes says such a thing about the dependence that *quantity* has on *extension*. For example, in *The World* Descartes says that the eternal truths such as *the whole is greater than its part* underwrite mathematics.[10] And, in a 27 May 1638 letter to Mersenne, Descartes says, in answer to a question Mersenne had posed in a previous letter:

> You ask whether there would be real space, as there is now, if God had created nothing. At first this question seems to be beyond the capacity of the human mind, like infinity, so that it would be unreasonable to discuss it; but in fact I think that it is merely beyond the capacity of our imagination, like the questions of the existence of God and the human soul. I believe that our intellect can reach the truth of the matter, which is, in my opinion, that not only would there not be any space, but even those truths which are called eternal—as that 'the whole is greater than its part'—would not be truths.[11]

We know that by *space* Descartes means corporeal substance—it is nothing short of *res extensa* itself.[12] Assuming that his remarks about the existence of space and that of the eternal truths of mathematics are related here, I will suggest in what follows that we read Descartes as saying that had God not created matter (that is, extension in length, breadth, and depth), there would be *no* mathematical truths.

I do not believe that Descartes is simply listing just one more thing that would not exist had God not created the *cosmos*—for, no doubt Mersenne could have figured out this on his own, namely, that had God not created *anything*, nothing would exist. What Mersenne seems to be fishing around for is an answer to the question of how far Descartes is prepared to go in identifying matter with extension. Although we do not have Mersenne's letter, the question seems to have been: wouldn't there still be empty space (void) had God not created anything? Since Descartes takes extension to be a real attribute, it must be the attribute of a real *something*.[13] As he will later put it: nothing cannot have any properties, which yields, 'as I have often said, *nothingness cannot possess any extension*'.[14] In this letter, although it is not perfectly clear, and is expressed rather sloppily, read in light of that *Principles* passage that claimed that quantity and number are unintelligible independent of extension (Cartesian matter), Descartes seems to be saying that God's not creating matter would be importantly connected to God's not creating mathematical truth. In other words, had God not created matter, there would be no mathematical truth. I will revisit this in Part IV and offer a more robust argument for taking Descartes's answer to Mersenne in exactly this way.

Descartes's remarks should remind us of Aquinas' position, briefly discussed in Part I. There, recall, Aquinas held there to be two distinct senses of 'division'. The first is a division of matter, which results in the numbers. The second is the division of the supra-natural, immaterial medium, which results in the plurality of reality (the forms).[15] A division of the supra-natural, immaterial medium, it was argued, was not enough to yield the conditions necessary for mathematics. Matter, it seems for Aquinas, as it seems now to be the case for Descartes (where matter is taken to be extension), grounds the very intelligibility or possibility of (and truth conditions for) mathematics. If there was only mind, or an immaterial substrate, there *could not be* quantity (or number). And since quantity is for Descartes only conceptually distinct from extension, if quantity or number is intelligible, there must be something more than solely an immaterial substrate—there must be a *material* substrate. For, quantity cannot be clearly and distinctly conceived apart from extension. But as I said, more on this line of reasoning in Part IV.

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[10] AT XI, 47; CSM I, 97.

[11] AT II, 138; CSMK III, 103.

[12] AT VIIIA, 45; CSM I, 227.

[13] AT VIIIA, 49; CSM I, 229–30.

[14] AT VIIIA, 50; CSM I, 231.

An initial response to the view that I am working out, understandably, may be to wonder whether Descartes *really* held it. (I now rehearse an objection that will inform Part IV.) For, certainly mathematical truths are true regardless of whether bodies exist. This is after all what Descartes says in the First Meditation. For, although we might doubt the truths discovered by the sciences of physics, astronomy, and so on (because we can doubt the existence of the objects they study), we are not in an obvious position to doubt pure mathematics, for 'arithmetic, geometry and other subjects of this kind, which deal only with the simplest and most general things, regardless of whether they exist in nature or not, contain something certain and indubitable'.[16] Of course, he *also* entertains in that same meditation the possibility of a supremely powerful deceiver, something that by the Fourth Meditation he characterizes as something that had involved him in a contradiction.[17] In other words, the First Meditation position about the supremely powerful deceiver not only turns out to be a position that he cannot now hold, but a position that he *could not have ever held*! Arguably, the First Meditation claim about mathematics has the same shelf life as the earlier claim about that supremely powerful deceiver. If so, Descartes's First Meditation position on mathematics may not be his actual position.[18] So, citing it will not be an obvious refutation of the view that I am entertaining here, which is: Descartes holds matter (*res extensa*) to be a necessary condition for both the intelligibility and the truths of mathematics. To move things forward, then, let's follow-up on the idea that modes of extension, or dimensions as Descartes sometimes calls them, are *determined* by measuring.

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[15] This comes primarily from the *Summa Theologica*. See the Part I discussion for the specific passages.

[16] AT VII, 20; CSM II, 14.

[17] AT VII, 54; CSM II, 38, AT VII, 62; CSM II, 43.

[18] Another contender here is Descartes's claim in the Fifth Meditation that there can be essences without corresponding particular instances or existents of those essences. He says: 'When, for example, I imagine a triangle, even if perhaps no such figure exists, or has ever existed, anywhere outside my thought, there is still a determinate nature, or essence, or form of the triangle which is immutable and eternal, and not invented by me or dependent on my mind' (AT VII, 64; CSM II, 44–5). I will have more to say about this sort of case in what follows.

Jill Buroker argues that Descartes's pursuit of a quantitative physics can be traced back to the *Rules*.[19] She contends that it is there that Descartes works out the connections between the concepts of *order*, *measure*, and *quantity*. There are four distinct, though related, kinds of scales: *nominal*, *ordinal*, *interval*, and *ratio*.[20] Buroker focuses on the last two. She suggests, for example, that for Descartes, interval scales were taken to measure what are sometimes called *intensive* properties—heat and colour are good examples. Such a scale, we might say, measures *intensity*. Emphasis here should be placed on the fact that interval scales (their units) are not additive. So, for example, if we have a bucket of water whose temperature is measured at 70 degrees (Fahrenheit), and another bucket of water whose temperature measures the same, and we combined the two into one bucket, the combined temperature would not now measure 140 degrees. Buroker also shows that, for Descartes, ratio scales were taken to measure what are sometimes called *extensive* properties—length and weight are good examples. The units of ratio scales *are* additive. So, for example, if we have a rod of iron measuring 6 feet in length and combine it (at one end, continuing along its length) with another rod measuring 6 feet, the rods combined *will* measure 12 feet in length. Buroker argues that for Descartes *only* those properties measurable by ratio scales are taken to be real properties (or modes) of bodies. These are the *dimensions* he talks about in the *Rules*. A consequence of his taking there to be a difference between interval and ratio scales, she contends, is that Descartes makes, albeit implicitly in the *Rules*, a distinction between what would later be called *secondary qualities* (measured by interval scales) and *primary qualities* (measured by ratio scales) of body.[21]

There is an interesting relation holding between ratio and interval scales—in fact, between all four kinds of scale. Logically or conceptually speaking, the interval scale is a necessary building block of the ratio scale; the ordinal scale is a necessary building block of the interval scale; and, the nominal scale is a necessary building block of the ordinal scale. It follows that the nominal scale is a necessary building block of the ratio scale. And so, if there is no nominal scale, there is no ratio scale. Simply put, nominal scales establish equality between things; ordinal scales establish an asymmetric order between things; interval scales establish the equality of the intervals holding between things; and ratio scales establish the equality of ratios (of measured quantity) holding between things.

In his *Philosophical Foundations of Physics*, Carnap states, 'The concepts of science, as well as those of everyday life, may be conveniently divided into three main groups: classificatory, comparative, and quantitative.'[22] Logically or conceptually speaking, the quantitative concept is 'built' out of the comparative concept, which in turn is built out of the classificatory concept. Here, the classificatory concept is connected to the nominal scale, the comparative concept is connected to the ordinal and interval scales,[23] and the quantitative concept is connected to the ratio scale. Given that the classificatory concept is connected to the nominal scale (such a scale *expresses* the concept), and the nominal scale is necessary for the building of the ratio scale, it follows that the classificatory concept is necessary for the building of the ratio scale. What I want to suggest is that the classificatory concept is expressed in Descartes's concept of *enumeration*.

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[19] Jill Vance Buroker, 'Descartes On Sensible Qualities', *Journal of the History of Philosophy*, vol. 29, no. 4 (October 1991), pp. 585–611.

[20] S. S. Stevens, 'On the Theory of Scales of Measurement', in *Philosophy of Science*, Chapter 8, edited by Arthur Danto and Sidney Morgenbesser (New York: The World Publishing Company, 1960), pp. 141–9. It was originally published in *Science* (1946), vol. 103, no. 2684.

[21] Buroker (1991), p. 597. For an earlier working out of the view that I am developing here, see Kurt Smith, 'Was Descartes's Physics Mathematical?' *History of Philosophy Quarterly*, vol. 20, no. 3, (July 2003), pp. 245–56.

Following Hempel, Carnap introduces two essential relations for the making of scales—e and l.[24] He says, 'we need a rule for defining an observable relation to equality', which he calls e, and 'a rule for defining a relation I will call "less than"', which he calls l. e is reflexive, symmetric, and transitive. l is neither reflexive nor symmetric, but it is transitive. It is, as he puts it, asymmetric and transitive.[25] The rules for each go as follows:

> Rule 1, for magnitude M, specifies an empirical relation e. The rule states that, if the relation e_M holds between objects a and b, the two objects will have equal values of the magnitude M. In symbolic form:
>
> $$\text{If } e_M(a, b), \text{ then } M(a) = M(b).$$
>
> Rule 2 specifies an empirical relation l_M. This rule says that, if the relation l_M holds between a and b, the value of the magnitude M will be smaller for a than for b. In symbolic form:
>
> $$\text{If } l_M(a, b), \text{ then } M(a) < M(b).$$[26]

Carnap puts the two together to begin the logical or conceptual building of our scales:

> 1) e must be an equivalence relation.
> 2) e and l must exclude each other. No pair of objects can be equally heavy and at the same time be so related that one is lighter than the other.
> 3) l must be transitive.

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[22] Rudolf Carnap, *Philosophical Foundations of Physics*, edited by Martin Gardener (New York: Basic Books, 1966), p. 51. Carl Hempel also claims this in his 'Fundamentals of Concept Formation in Empirical Science', *The International Encyclopedia of Unified Science* (Chicago: The University of Chicago Press, 1952), vol. 2, no. 7.

[23] Stevens (1960) appears to argue that the idea of quantity begins with (or at the level of) the interval scale (p. 146).

[24] Carnap (1966), Chapters 5–6. He uses the capital E and L, but since I use E to stand for the enumeration of extended things, and lower case e and l to stand for particular members of E, to avoid confusion I will use lower case italic *e* and *l* for Carnap.

[25] Ibid., p. 55.

[26] Ibid., p. 63.

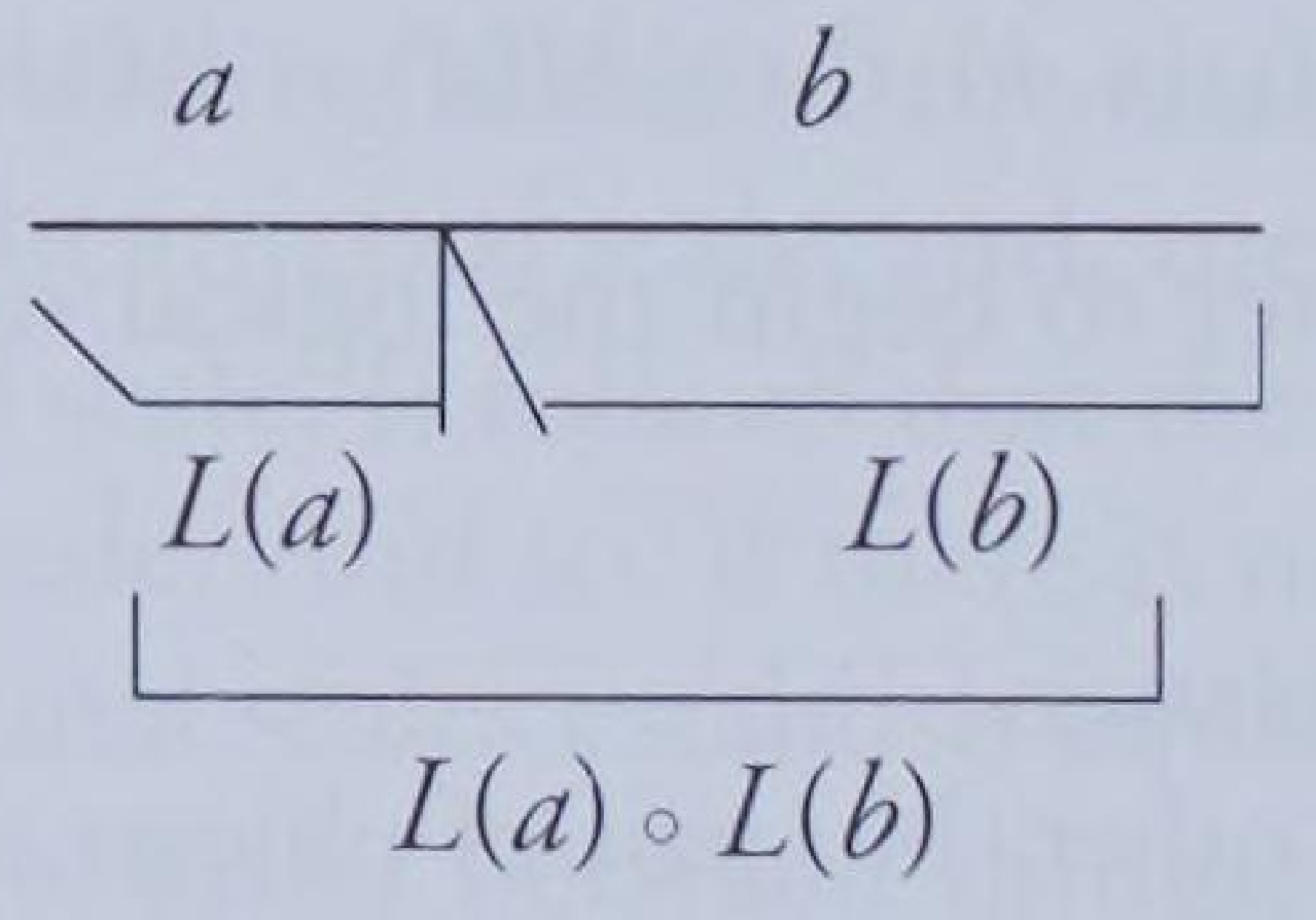
> 4) For any two objects *a* and *b*, one of the three following cases must hold. (Actually, it is sufficient to say that at least one holds. It then follows from the other conditions that exactly one will hold.)
>
>   a. *e* holds between the two objects.
>   b. *l* holds between *a* and *b*.
>   c. *l* holds between *b* and *a*.
>
> In other words, any two objects *a* and *b* that have weight are either equal in weight, or *a* is lighter than *b*, or *b* is lighter than *a*.[27]

There are three other rules that Carnap introduces. Rule 3 tells us how to assign an arbitrary zero, 0, to the scale; Rule 4, which he notes is sometimes referred to as the 'rule of the unit', tells us how to assign the unit of the scale, usually 1; Rule 5 tells us how to establish the conditions under which two *differences* of values of magnitude are equal.[28] From the above we could build at the very least a scale that measured temperature—an interval scale.

In order to move to the conceptual level of the ratio scale we must work up 'what is called the principle of addition, or "additivity" '.[29] Warning about earlier 'clumsy' attempts at working this up, where his predecessors had failed to keep the 'adding' of numbers distinct from the 'adding' of things, Carnap says:

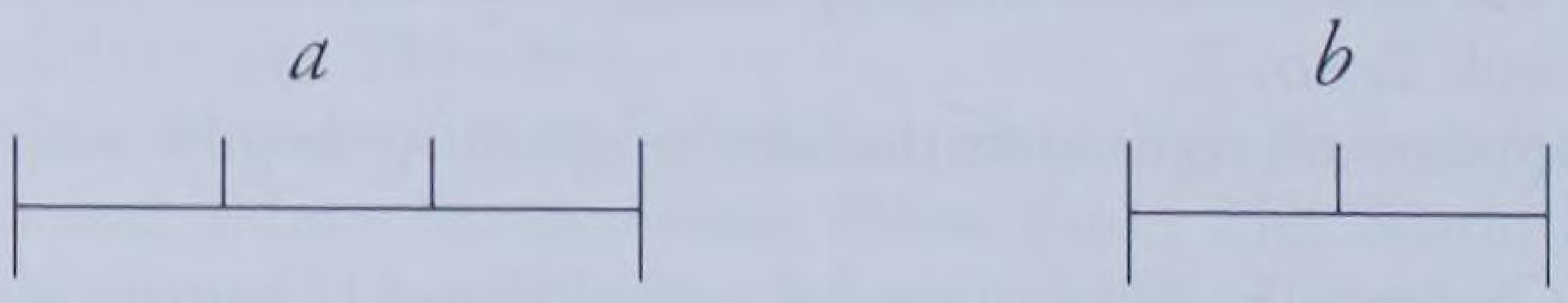
> It helps us to keep this distinction in mind if we follow Hempel (who has written at length about extensive magnitudes) in introducing a special symbol, a small circle '∘', for the physical operation of joining. This provides a much more satisfactory way of symbolizing the additive rule for length:
>
> $$L(a \circ b) = L(a) + L(b).$$

The combination of lengths can be diagrammed:



Compare what Carnap says above to what Descartes says in Rule Eighteen of the *Rules*:[30]

> If addition . . . is to be used, we conceive the subject in the form of a line, or in the form of an extended magnitude in which length alone is to be considered. For if we are to add line *a* to line *b*,



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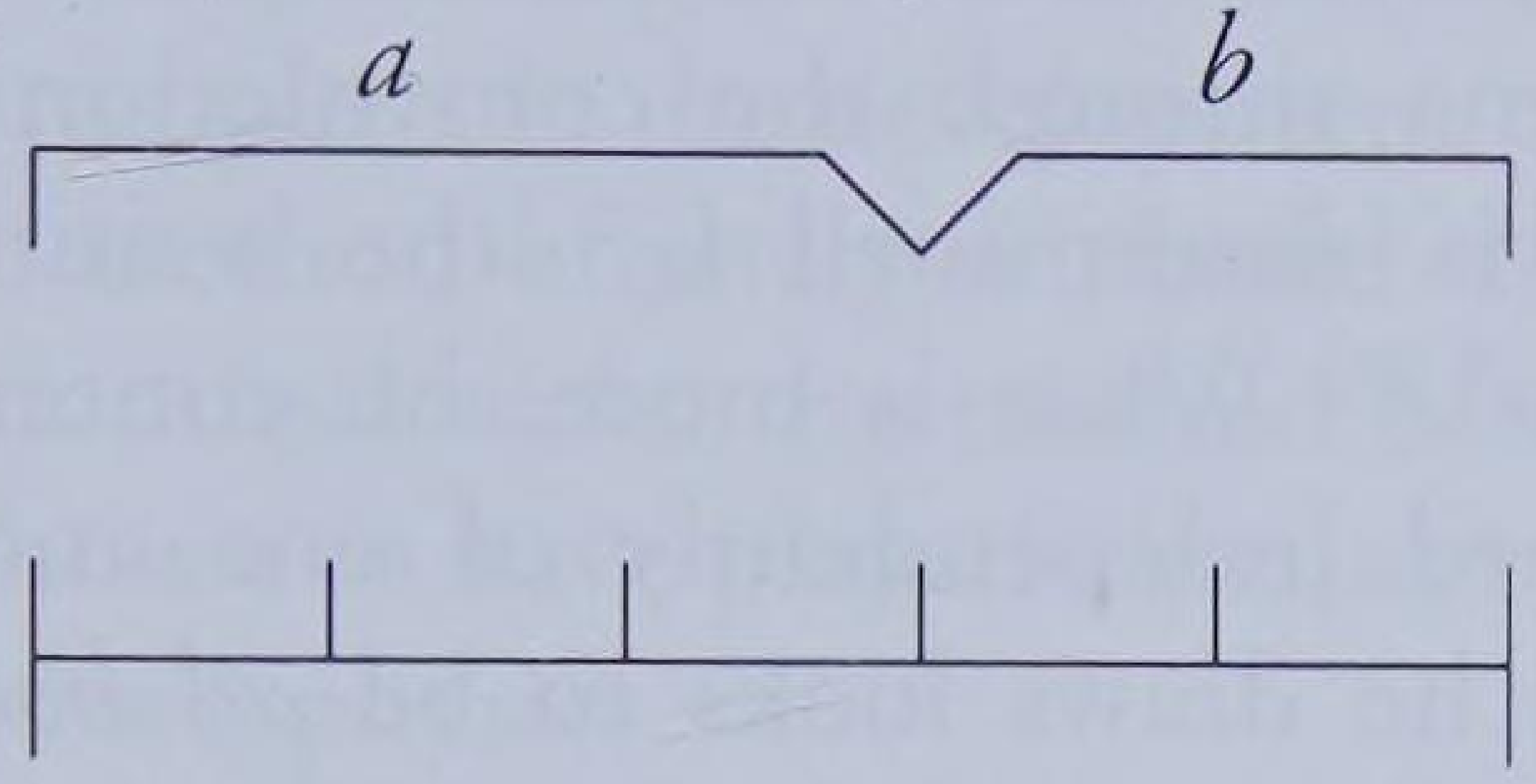
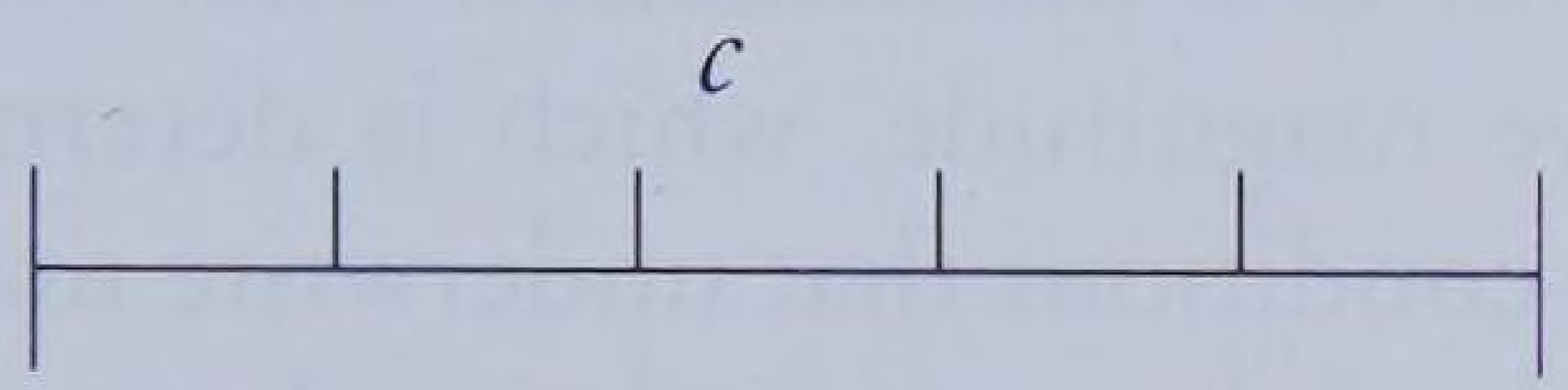
[27] Carnap (1966), p. 57.
[28] Ibid., p. 645.
[29] Ibid., p. 71.
[30] AT X, 464; CSM I, 73.

we add the one to the other, in the following way,



and the result is *c*:



Although Descartes's view may look relatively simple compared to Carnap's, I believe that a more rigorous study of Descartes's *Rules* would show that the simplicity is only apparent.

◆

Generally, we start then with a set of objects to be investigated. Using our equivalence relation *e*, and comparing the objects under investigation, we would establish a partition, or enumeration. This is a *nominal* scale. Now, suppose that in using *l*, our asymmetric and transitive relation, we compare the members of the various classes of the enumeration, discovering an asymmetry to hold between them. We could order the classes according to the discovered asymmetry. This would be an *ordinal* scale. Moving further, if we could establish an equality of the intervals separating the ordered classes, we would establish an *interval* scale. Lastly, if we could establish an equality of ratios holding between the intervals that separate the ordered classes, we would establish a *ratio* scale.

As Descartes rightly points out about the isomorphism between the 'additive' properties of bodies and the additive property of numbers:

> Thus it is certain that whatever differences of proportion obtain in other subjects (the objects being studied), the same differences can also be found to hold between two or more extensions.[31]

Since the units of a ratio scale are genuine *quantities*, or *quantitative magnitudes*, they are additive. What this means is that the properties of bodies measured by (and thus represented by the quantities of) ratio scales are subject to mathematical treatment. This in part is what is conceptually underwriting a *quantitative* physics. And, as Buroker has argued (and I have further argued here), it is something that Descartes seems to be working out in the *Rules*.

Yves Gingras has argued that there is a difference between a quantitative physics and a mathematical physics.[32] And, he has gone as far as to argue that, although quantitative, Descartes's physics was not a mathematical physics.[33] He says that a quantitative physics is determined by 'the production of numbers for measuring phenomena through the construction of a metric',[34] whereas a mathematical physics is determined by 'the writing of abstract geometric or algebraic formulations'.[35] What is more, he contends that the two can be developed and maintained independently of one another. Although he may be right, the distinction he draws looks to be *ad hoc*. And, the determining criterion he uses for 'mathematical' looks to be too narrow. I would think that a quantitative physics *is* a mathematical one, for the simple reason that a quantity or quantitative magnitude, which is determined by a ratio scale, is underwritten by the very conditions that underwrite mathematics—the additive conceptual building block of the ratio scale is what mathematicians now call a *group*. Thus far, we have only mentioned this feature in terms of Carnap's rule of additivity, and in Part III we will look very carefully at it. But, if I am right, and a quantitative system of measurement is determined by the very conditions that underwrite mathematics (in particular the conditions underwriting the operations of addition and multiplication), then for the very same reasons that mathematics is 'mathematical', so too is a quantitative system of measurement. Surely, Gingras will not say that what makes mathematics mathematical is *simply* the writing of fancy formulae. So, why think that this is what makes a physics mathematical?

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[31] AT X, 447; CSM I, 62.

[32] Yves Gingras, 'What Did Mathematics Do to Physics?' *History of Science*, vol. 39 (2001), pp. 383–416.

At this point we could forward the following argument: given that a quantitative physics is a mathematical physics, and that Descartes's physics is a quantitative physics, it will follow that Descartes's physics is a mathematical physics. I suspect, however, that much more would have to be done to convince Gingras and others that the argument is sound. By the end of our final chapter, we should have enough on the table to conclude, without any doubt, that the above argument is sound, and consequently that not only Descartes's, but Leibniz's physics was a mathematical physics.

At the start of this chapter on analysis, I promised that we would see how the method of analysis, understood as a method of collection and division (or enumerating), was internally connected to Descartes's theory of clarity and distinctness. Although there was certainly more that could have been said on this, enough was brought to light in the above to make the connection. I also promised that we would see how the enumeration underwrites the conditions necessary for the concepts of *quantity* and *measurement*. It was shown how *quantity* is determined by the ratio scale, and how it is that the ratio scale has, as one of its necessary building blocks, the nominal scale. Since the enumeration is a nominal scale, we see how the enumeration worked for Descartes (and Leibniz and others) as the necessary condition underwriting the concepts of *quantity* and *measurement*. Lastly, I promised that by the chapter's end it would be made clear what part analysis played in the development of a mathematical physics. Since analysis is the method of constructing the enumeration (the nominal scale), and knowing what part the latter plays in the development of a mathematical physics, we can see what part the method of analysis played. If a quantitative physics is impossible without the nominal scale, the method of analysis, which is used to produce such a scale, is shown to have played a major role in the development of a quantitative (mathematical) physics.

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[33] Ibid., p. 387.

[34] Ibid., fn. 10, pp. 407–8.

[35] Ibid.

PART III

SYNTHESIS

8

Synthesis and System Building

In the Second Replies, Descartes briefly works out the main argument of the *Meditations* in synthetic form.[1] He begins with establishing definitions and postulates, moves to expressing axioms or common notions, and then to proving propositions. But, as Curley points out, the above may have misled commentators.[2] For, if the conversation with Burman is reliable, Descartes takes himself to have also written the *Principles* in synthetic form, despite the fact that there is no formal geometric structure, categorization of axiom, postulate, proposition, and so on—something that Curley says threw even Spinoza, which in part accounts for Spinoza's later reworking of the *Principles* in the classical synthetic form.[3] So, either Descartes is not perfectly clear on what synthesis is, or he simply is not adhering to the method as classically understood. What I want to show in this part is that for Descartes, Leibniz, and others of the period, the method of synthesis began to take on a new look, which strayed from the classical geometrical version.

The method, I contend, would come to be understood primarily in terms of combinatorics. As Leibniz unambiguously puts it, 'There are two methods, synthesis or the art of combinations, and analysis.'[4] The idea seems to have been that we perform synthesis whenever we unite elemental concepts in the enumeration in the making of others, which in turn can be used in the making of others. The pay-off is that the method of synthesis emerges as a method for the making of *equations* in physics. What also emerges from this method, though Leibniz and others were only beginning to flirt with it, is the insight into the concept of a *group*—in connection to combinatorics, a *permutation group*—a concept that expresses the conditions underwriting a mathematical system. Ancestors of the concept also appear to be springing up in the logical calculi, and in the designs of calculating machines. Towards the close of this part, it should be clear how one result of the method of synthesis—the making of equations—is connected to the concepts of *quantity* and *measure*. Here, the permutation group concept, though not fully expressed by seventeenth-century mathematicians (though almost fully expressed by those in the eighteenth century), turns out to provide what Carnap called the *principle of addition*, or *additivity*, a necessary building block of the ratio scale. By end of Part III, then, it should be clear to the reader what part synthesis played in the development of a mathematical physics, and in what sense certain philosophers of the period understood matter to be connected to mathematics.

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[1] AT VII, 160–70; CSM II, 113–20.

[2] Curley (1986), p. 155. He also deals with Spinoza's synthetic rendering of the *Principles* in 'Spinoza as an Expositor of Descartes,' in *Speculum Spinozanum*, edited by S. Hessing (London: Routledge & Kegan Paul, 1977), pp. 133–42.

[3] Curley (1986), p. 155.

[4] Gottfried Leibniz, *Leibnizens mathematische Schriften*, C. I. Gerhardt (ed.), 7 vols. (Berline, 1846–1860). This is found in vol. 1, p. 557.

One sense in which Descartes and others of the period understood matter and mathematics to be connected, a sense on which I will primarily focus, is expressed by the conditional:

> If matter exists, mathematics is intelligible.

Here, the conditional is taken to be telling us that the existence of matter (Cartesian extension) is sufficient for establishing the possibility of mathematics. To be sure, there are those who believe that the existence of matter and the possibility of mathematics are two very different things. So, for such critics one can deny the existence of matter and yet assert that mathematics is intelligible (or possible). But, in so doing, they have rendered the above conditional (trivially) true. So, such critics should find no trouble accepting the conditional. Other critics may say that the conditional is trivially true since its consequent is a necessary truth. For lack of a better name, I will call such accounts of the truth of the above conditional *negative*. What I propose in this and the chapters to follow is a *positive* account, where what we will learn is how thinkers of the period took the existence of matter to be sufficient for establishing the possibility of mathematics, and how it is that, if the intelligibility or possibility of mathematics were denied, it would be logically required to deny the existence of matter.

NATURAL VERSUS COMPOSITE UNITY

In the Sixth Replies, Descartes draws a distinction between two sorts of unity: unity *by nature* and unity *by composition*. Consider, he says, someone who believes that his mind and body are attributes of the same substance. Such a person walks around and thinks, and in attributing both the moving and thinking to the same subject (that is, to himself), he believes himself to be a substance whose nature includes extension (of which movement is a modification) and thinking. Descartes says:

> Because, however, he is accustomed to attribute many different properties to one and the same subject without being aware of any connection between them, he may possibly be inclined to doubt, or may even affirm, that he is one and the same being who thinks and who moves from place to place. Notice that if we have different ideas of two things, there are two ways in which they can be taken to be one and the same thing: either in virtue

> of the unity or identity of their nature, or else merely in respect of unity of composition. For example, the ideas which we have of shape and of motion are not the same, nor are our ideas of understanding and volition, nor of those of bone and flesh, nor are those of thought and of an extended thing. But nevertheless we clearly perceive that the same substance which is such that it is capable of taking on shape is also such that it is capable of being moved, and hence that which has shape and that which is mobile are one and the same in virtue of a unity of nature. Similarly, the thing that understands and the thing that wills are one and the same in virtue of a unity of nature . . . But now the question is whether we perceive that a thinking thing and an extended thing are one and the same by a unity of nature. That is to say, do we find between thought and extension the same kind of affinity or connection that we find between shape and motion, or understanding and volition? Alternatively, when they are said to be 'one and the same' is this not rather in respect of unity of composition?[5]

The first thing on which to focus is Descartes's saying that the above person does not conceive any *connection* between thinking and extension, which are the properties that this person attributes to himself as subject. This is key. For, in terms of the enumeration, such a person cannot ever be clear about such a nature—he fails to see the relation that *unites* thinking and extension. This is different, Descartes says, from shape and motion. For, here one can (and presumably does) conceive the connection—both entail (the concept of) extension, for instance, which is another way of saying that one could not conceive a shaped or moving thing without also conceiving of a thing that is extended. Things that we believe to be unified, but where the relation between the elements that constitute them cannot be conceived, are composite unities. The human being is a prime example. This, he says, is in contrast to things unified by nature, examples of which are shape and motion, which are modes of extension (we might add the logical point—shape and motion entail extension), and understanding and volition, which are modes of thinking (and here understanding and volition entail thinking).

Notice how the above aligns with how the master enumeration (determined by the method of analysis) divides the simple natures. Those entities falling under T (the class of thinking things) are unified by nature; those entities falling under E (the class of extended things) are also unified by nature. But, whenever entities falling under both are combined, the sort of unity they express is unity by composition—and *not* unity by nature. This does not deny a genuine unity of thought and extension in the human being, but it does deny our ever clearly and distinctly perceiving that unity. This is echoed in his conversation with Burman, where he says:

> You have a clear conception of corporeal substance, and you also have a clear conception of thinking substance as distinct from, and incompatible with, corporeal substance. In view of this, you would be going against your own powers of reasoning in the most

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[5] AT VII, 423–4; CSM II, 285–6.

> absurd fashion if you said the two were one and the same substance. For you have a clear conception of them as two substances which not only do not entail one another but are actually incompatible.[6]

As noted in Parts I and II, 'entailment' here is to be understood in light of the conceptual structure expressed by way of the enumeration. And so, when Descartes says that extension does not entail thinking, and vice versa, he is noting something that is clearly shown by way of the enumeration—the sets of thinking and extended things are mutually exclusive, and one is not contained in the other, which is another way of expressing the idea that one does not entail the other.

In *Comments on a Certain Broadsheet* Descartes again mentions two sorts of unity, this time associating the first with composite entities, and the second with simple entities. He says:

> A composite entity is one which is found to have two or more attributes, each one of which can be distinctly understood apart from the other. For, in virtue of the fact that one of these attributes can be distinctly understood apart from the other, we know that the one is not a mode of the other, but is a thing, or attribute of a thing, which can subsist without the other. A simple entity, on the other hand, is one in which no such attributes are to be found. It is clear from this that a subject which we understand to possess solely extension and the various modes of extension is a simple entity; so too is a subject which we recognize as having thought and the various modes of thought as its sole attribute. But that which we regard as having at the same time both extension and thought is a composite entity, namely a man—an entity consisting of a soul and a body.[7]

A composite entity here looks to be that which in the Sixth Replies he said is a unity by composition. A simple entity looks to be that which he said is a unity by nature. In both texts, Descartes proposes the human being as an example of the former, and modes of extension (shape and motion) as examples of the latter.

In the Sixth Meditation, Descartes tells us that corporeal nature 'is the subject-matter of pure mathematics'.[8] He later remarks that bodies must 'possess all the properties which I clearly and distinctly understand, that is, all those which, viewed in general terms, are comprised within the subject-matter of pure mathematics'.[9] Now, the figures studied by geometers are not *things* or *substances* in corporeal reality, but rather, as he tells Gassendi in the Fifth Replies, 'the geometrical figures are considered not as substances but as boundaries within which a substance is contained'.[10] In the language of the *Principles*, I will suggest that Descartes is claiming that mathematics (including geometry) studies things that are modally distinct. In terms of the enumeration, a mathematical physics works within the system of dimensions of body, the enumerations of E, which is the system of modes determined by ratio scales.

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[6] AT V, 163; CSMK III, 345.

[7] AT VIIIB, 350–51; CSM I, 299.

[8] AT VII, 74; CSM II, 51.

[9] AT VII, 80; CSM II, 55.

[10] AT VII, 381; CSM II, 262.

Why Descartes holds that mathematics deals *solely* with matter (extended substance) and not with thinking substance is not perfectly clear. Like Aquinas (discussed in Parts I and II), Descartes holds the underlying condition for the *intelligibility* of quantity, such as the quantity *ten feet*, to be matter.[11] Even so, he does not see physics and mathematics as being coextensive. For example, to Burman he says:

> The only difference is that physics considers its object not just as a true and real entity, but also as something actually and specifically existing. Mathematics, on the other hand, considers its object merely as possible, i.e., as something which does not actually exist in space but is capable of so doing.[12]

Things that are clearly and distinctly perceived, he says elsewhere, are true entities.[13] So, in terms of the master enumeration, body regarded as sized, shaped, moving, and so on, is a true entity. What Descartes seems to have in mind in the above distinction between physics and mathematics is our considering *this particular* existent body, with *these particular* dimensions, located *here* and not there, versus our considering *a* body, with dimensions arbitrarily assigned to it, not necessarily existing here or there. Physics deals with the former; pure mathematics with the latter.[14] Notice that both deal with extension. Recall from Part I that Aquinas drew a similar distinction. There, we saw that Aquinas took mathematics to deal with objects that depend for their existence on matter, but nevertheless *could* be understood apart from (or without reference to) *sensible* matter. The latter Aquinas associates with particular bodies, which, he says, are the objects of physics.

Put in the language of the Cartesian texts considered thus far, a mathematical physics deals with an object that is a unity by nature, a simple entity, described within the context of the master enumeration as the enumerations of E, the system of dimensions or modes determined by ratio scales. Synthesis, I contend, can be understood as a method of combining those dimensions (represented in the enumeration as classes), the result of which are equations, which describe how bodies (and their modes) actually stand to one another. Let's now turn to looking at several examples of equation making.

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[11] AT VIIIA, 44; CSM I, 226.

[12] AT V, 160; CSMK III, 343.

[13] AT V, 160; CSMK III, 343.

[14] In discussions with Dan Garber at the National Endowment for the Humanities (NEH) Summer Institute, 'The Intersection of Theology, Philosophy, and Science in the Seventeenth Century', held at the University of Wisconsin, Madison, Garber brought to light another difference between physics and mathematics, namely physical explanations on Descartes's view included reference to the *nature* of objects, whereas mathematics simply described phenomena. Descartes complains, for instance, that Galileo's mathematical 'explanations' are really not explanations at all, for they do not explain *why* a body does what it does. Another difference that Garber has brought to light is that whereas mechanics or applied mathematics takes into consideration gravity, for instance, mathematical philosophy does not.

SYNTHESIS AND EQUATION MAKING

In Part II we considered how dimensions are represented in the enumeration. First, recall, we began by considering E, the class of extended things, whose members a, b, c, and so on, are determinate bodies:

$$E = \{a, b, c, \ldots, x\}.$$

An enumeration of a *mode* of extension is a collection and division of the members of E done by recasting them in terms of the mode. So, let's again consider size (Z), which is determined by some system of measure:

$$\begin{array}{lcccccc} & 1 & 2 & 3 & 4 & ,\ldots, & N \\ Z = \{ & \{f, g, k\}, & \{a, j\}, & \{b, c, r, o\}, & \{m, q\}, & \ldots, & \{x, y, z\}\}. \end{array}$$

The relation we used to make this partition was *x is the same size as y*, where objects having 1 unit of size belong to the class whose number is **1**, objects having 2 units of size belong to the class whose number is **2**, and so on. Thus, f, g, and k have 1 unit of size and are members of the class whose number is **1**, a and j have 2 units of size and are members of the class whose number is **2**, and so on. Since this is an enumeration determined by a ratio scale, the units are additive. Thus, if we combined the magnitudes of a and m, which are 2 and 4 respectively, the resulting magnitude will be 6 units of size. Also, the ratio scale guarantees that the *ratio* holding between the magnitude of f and a, which here is 1:2, is equal to that holding between a and m, which is 2:4. As I noted in Part II, these classes would have more members than I show here. I am simply providing the above for the purpose of explanation.

Now, let's recast the members of E in terms of the dimension of speed (V). So, we measure the objects according to this particular system of measure and produce the following partition of E:

$$\begin{array}{lccccc} & 1 & 2 & 3 & ,\ldots, & N \\ V = \{ & \{r, h, k\}, & \{n, i, w, q\}, & \{a, m, v, s\}, & \ldots, & \{l, c\}\}. \end{array}$$

The relation we used to make this partition was *x has the same speed as y*. The measured speed of n is 2 units of speed, the measured speed of a is 3 units of speed, and so on. Since a ratio scale determined this enumeration, the same properties holding for the magnitudes of Z also hold of V. Notice that V enumerates E differently in this case than Z enumerates E. Since a has 2 units of size, in Z we find it in the class designated by **2**; since a has 3 units of speed, in V we find it in the class designated by **3**. The point to keep in mind is that each dimension, or mode of extension determined by some system of measure (a ratio scale), will partition E.

Ernest Nagel argues that in physics there is an important connection between measuring and the constructing of mathematical equations:

> If the search for mathematical equations is the aim of physics, other activities such as experiment, classification, or measurement are subservient to this aim and are to be understood only in relation to it. Consequently, if we inquire why we measure in physics, the answer will be that if we do measure, and measure in certain ways, then it will be possible to establish the equations and theories which are the goal of inquiry.[15]

We find that Nagel's remark about measuring 'in certain ways' echoes Carnap's (and Hempel's) insight into the need for a principle of addition. In other words, the scale which we use to measure must include in its construction *additivity*—its units must be additive. This feature makes the system of measurement subject to mathematical treatment. Since the only scales that possess this property are ratio scales, the scales of measurement involved must be ratio scales.

In *Principles*, Part II, Article 36, Descartes discusses what I think is an example of equation making. He says:

> Admittedly motion is simply a mode of the matter which is moved. But nevertheless it has a certain determinate quantity; and this, we easily understand, may be constant in the universe as a whole while varying in any given part. Thus if one part of matter moves twice as fast as another which is twice as large, we must consider that there is the same quantity of motion in each part; and if one part slows down, we must suppose that some other part of equal size speeds up by the same amount.[16]

The *total* quantity of motion of bodies in the universe is constant. Thus, if one body begins to move faster, some other body must begin to move proportionately slower—otherwise, if all bodies remained as they were and a body all of a sudden began moving faster, the total quantity of motion in the universe would increase, which is impossible according to Descartes. In this passage, he specifically relates the *size* and *speed* of a body to the quantity of motion, so that if the quantity of motion of bodies a and b are equal, then if a is twice the size of b, b's speed must in turn be twice that of a. In relating size and speed in this way, he seems to understand quantity of motion as follows:

$$\text{size} \times \text{speed} = \text{quantity of motion.}$$

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[15] Ernest Nagel, 'Measurement', in *Philosophy of Science*, Chapter 7, edited by Arthur Danto and Sydney Morgenbesser (New York: The World Publishing Company, 1960), pp. 121–40. This is a selection from *Erkenntnis*, II Band, Heft 5, 1932, pp. 313–33. The quotation comes from p. 123 of *Philosophy of Science* (1960).

[16] AT VIIIA, 61; CSM I, 240.

Let the dimension *quantity of motion* in the enumeration of E be designated by Q. Thus, where *size* and *speed* are Z and V respectively, the equation is:

$$Z \times V = Q.$$

Now, recall Z and V:

$$Z = \{\overset{1}{\{f, g, k, i\}}, \overset{2}{\{a, j\}}, \overset{3}{\{b, c, r, o\}}, \overset{,\ldots,}{\ldots}, \overset{N}{\{x, y, z\}}\},$$

$$V = \{\overset{1}{\{r, h, k\}}, \overset{2}{\{n, i, w, q\}}, \overset{3}{\{a, m, v, s\}}, \overset{,\ldots,}{\ldots}, \overset{N}{\{l, c\}}\}.$$

Suppose that we wanted to calculate the quantity of motion of body a. If we look at the enumeration of size, Z, a is found in the class designated by **2**. We know that this is to be understood as a's having 2 units of size. And, if we look at the enumeration of speed, V, a is found in the class designated by **3**, which we know is to be understood as a's having 3 units of speed. Given the equation above, $Z \times V = Q$, where $a_z = 2$ and $a_v = 3$, we get:

$$2 \times 3 = 6,$$

which should be understood as telling us that a has 6 units of quantity of motion. And so, if we were to look at the enumeration of the quantity of motion we should find a in the class designated by **6**. The enumeration might look something like this:

$$Q = \{\overset{1}{\{k, \ldots\}}, \overset{2}{\{i, \ldots\}}, \overset{,\ldots,}{\ldots}, \overset{6}{\{a, \ldots\}}, \overset{,\ldots,}{\ldots}, \overset{N}{\{x, \ldots\}}\}$$

So, assuming for simplicity's sake that there are only two bodies in the universe, a and b, where we know that the quantity of motion of a is m units of quantity of motion, and the quantity of motion of a and b is equal, then we know that b's quantity of motion is also m. Given Descartes's equation (some might want to call this a definition)[17] of quantity of motion, if it turns out that the size of a is twice that of b's, then in order to preserve their being equal in respect to the quantity of motion, *b*'s speed will have to be twice that of *a*'s. An equation in Cartesian physics, then, might look like this:

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[17] I am indebted to Emily Grosholz for bringing this to my attention. Moreover, she also made it clear to me that the proper equation to be considered is what I present as $a_z a_v = b_z b_v$ and not 'size x speed = quantity of motion'. The former, she argues, is an equation, the latter a definition. Even so, in the context of my discussion, I see no problem in referring to the latter as an equation. I should also mention that Harré (1964) would seem to count this sort of equation as part of what he calls a 'reticular' theory (pp. 8–18).

$$a_z a_v = b_z b_v.$$

This, it seems to me, is what Descartes is getting at in the above *Principles* passage.

Now, let's push this a bit further. Suppose that we know the size and speed of *a*, and yet only the size of *b*. Given the above equation, we could easily calculate *b*'s speed. What is more, we would expect observations to confirm that the equation holds pre- and post-collision. So, where pre-collision (indicated here with single $'$) is:

$$a_z' a_v' = b_z' b_v',$$

we should observe post-collision (indicated here with double $''$), provided $a_z'' a_v'' = a_z' a_v'$, that:

$$a_z'' a_v'' = b_z'' b_v''.$$

Now, suppose that post-collision, we discover that:

$$a_z'' a_v'' \neq a_z' a_v',$$

where measurements show that the quantity of motion of $a_z'' a_v''$ is half that of $a_z' a_v'$. Given Descartes's conservation principle, it follows that the quantity of motion of *b* has increased by a proportion of 2 (which is what he says in the above *Principles* passage). This would involve either an increase in *b*'s size, or speed, or both. If, for example, *b*'s size pre-collision, that is b_z', is identical to that of post-collision, b_z'', then we know that *b*'s speed is what has increased. And, ideally we would expect *b*'s increase to be the proportional loss from *a*. If *b* did not increase in size, we might infer that the loss we will find in *a* will be with respect to *a*'s speed (for it did not give any size to *b* in the collision). And, if we find that *a*'s quantity of motion post-collision is strangely not aligned with what the equation predicts, then an examination of *a* post-collision should show that *a* did lose size (as well as speed) in the collision with *b* after all; but instead of giving to *b* some of its size, it lost a fragment of itself (which is the lost size of *a*), which is now a third body *c*, which will have some of *a*'s (and perhaps even some of *b*'s) initial speed. The equations, then, become instrumental in helping the physicist observe—in the unexpected case where $a_z'' a_v'' \neq b_z'' b_v''$, for example, and yet $b_z'' b_v'' = b_z' b_v'$, it suggested to the physicist that there is a third body that needed to be found, which if found should meet:

$$a_z'' a_v'' + c_z c_v = b_z'' b_v''.$$

Notice that an equation, like size x speed = quantity of motion, shows us how the various enumerations of dimensions of *E* (that is, how the modes of extension) are related. As in the early *Geometry*, an equation is *clear* whenever it explicitly shows the relations holding between the equation's parts—where the relations are mathematical operations. This way of making an equation is the method of synthesis. We 'put together' the dimensions of size and speed in order to produce a new dimension: quantity of motion. Analysis, of course, would be our starting with quantity of motion and trying to figure out what more simple elements make it up. So, ideally, we would arrive at an understanding that quantity of motion can be 'broken up' into the dimensions of size and speed, where the relation that unifies them (to form the dimension of quantity of motion) is the mathematical operation of multiplication. This would meet Descartes's criteria of being both clear and distinct with respect to the idea of the quantity of motion (here expressed as an equation). In this case, the idea would be clear because we see the elements of the idea and how they are internally related, and the idea would be distinct because it includes only those elements found in the class of the enumeration denoted by *E*.

Another example of the method of building equations (which I am suggesting is synthesis) from the enumerations of *E* and observation is found in the *Rules*. About knowledge of the magnet, he says:

> But take someone who thinks that nothing in the magnet can be known which does not consist of certain self-evident, simple natures: he is in no doubt about how he should proceed. First he carefully gathers together all the available observations concerning the stone in question; then he tries to deduce from this what sort of mixture of simple natures is necessary for producing all the effects which the magnet is found to have. Once he has discovered the mixture, he is in a position to make the bold claim that he has grasped the true nature of the magnet, so far as it is humanly possible to discover it on the basis of given observations.[18]

I take it that once one gets an idea of the simple natures involved in producing the magnet's effects (attracting pieces of iron, for example), one then should begin arranging the simple natures, relating them by way of mathematical operations. If one could produce an equation that worked in describing (and now predicting) the magnet's effects, one would have discovered, as far as is humanly possible, the true nature of the magnet.

Much of the debate in the period over the viability of Cartesian physics, I contend, is directly connected to concerns over whether Descartes had established *the* enumeration of dimensions of *E*. The concerns would include not only those about his having established the primitive or atomic categories but about his having established the complex categories—for example, whether the complex structure included all the simple natures necessary to describe the complex dimension, whether the equations (or definitions) of the latter were well structured, whether the parts (simple natures involved) were correctly related by way of mathematical operations, and so on. Even so, the debate is not very obvious and some work must be done to uncover it. But once it is uncovered, the debate jumps right out and it is difficult to ignore (in fact, one finds it at every turn).

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[18] AT X, 427; CSM I, 49–50.

We find Leibniz, for instance, concerned over Descartes's enumeration of corporeal substance. In a letter to Jacob Thomasius (April 1669), he writes:

> Primary matter is mass itself, in which there is nothing but extension and antitypy or impenetrability. It has extension from the space it fills. The very nature of matter consists in its being something solid and impenetrable and therefore mobile when something else strikes it, and it must give way to the other.[19]

Here, Leibniz rejects Descartes's claim that the nature of (primary) matter is solely extension in length, breadth, and depth. Matter, as opposed to space, Leibniz says, is also *impenetrable* or *solid*. Thus, body—that is, impenetrable extension—represented in the enumeration as E, should be taken to represent what he refers to as *mass* and not simply extension in length, breadth, and depth. So, for Leibniz E is the set of things that are impenetrable and extended as opposed to the set of things that are solely extended.

The Swiss mathematician Leonhard Euler, born towards the end of Leibniz's lifetime, and student of Jean Bernoulli, is found continuing the debate (which shows, I think, that the issues here were far from settled). I mention Euler not only because of his role in the history of mathematics (which we will consider shortly), but because we find him specifically worried about the issues now under discussion, in particular referring to Descartes and Leibniz in his discussions of these issues. Familiar with the philosophical works of both Descartes and Leibniz (as well as with their available mathematical works).[20] Euler argues, for example, that even though Descartes and the Cartesians were right to think that extension in length, breadth, and depth were the three dimensions needed to describe a geometrical solid, the nature of body cannot be *simply* its extension in length, breadth, and depth.[21] One addition that should be made to this Cartesian idea of body, he says, is '*impenetrability*, the impossibility of being penetrated by other bodies'.[22] In another letter, Euler defines body thus: '*a body*

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[19] Gottfried Leibniz, 'Letter to Jacob Thomasius' (1669), in *Leibniz: Philosophical Papers and Letters*, 2nd edn., translated by Leroy Loemker (Dordrecht: Reidel, 1969), p. 95.

[20] Leonhard Euler, *Letters of Euler*, 2 vols, translated by David Brewster (New York: J & J Harper, 1833). Euler discusses both Leibniz and Wolff (in letters included in vol. I). He specifically discusses Leibniz in Letter LXXXIII, titled 'Examination of the System of pre-established Harmony. An Objection to it', 9 December 1760, pp. 277–9.

[21] *Letters of Euler*, vol. I, Letter LXIX, 'Nature and Essence of Bodies; or Extension, Mobility, and Impenetrability of Body', 21 October 1760, pp. 233–5; Letter LXX, 'Impenetrability of Bodies', 25 October 1760, pp. 236–8.

[22] Letters of Euler, p. 235.

is an impenetrable extension'.[23] These are not the only remarks that would echo Leibniz's. Euler argues, for instance, that impenetrability and the conservation principle conceptually yield the idea that when bodies interact, the calculation of the result must reflect what he calls the *Principle of Least Possible Action*.[24] In other words, one should consider quantitative changes in the least number of dimensions, represented as parts of an equation. My point is that the concern is not limited to Leibniz (though I will focus almost exclusively on him here): it is a debate involving many living in the seventeenth and eighteenth centuries, extending well into the nineteenth century.

Another issue of concern was the enumeration of speed. Speed is the distance (D) an object travels over some period of time (T). It is the ratio of $D{:}T$. In terms of Descartes's enumeration, the equation (or definition) is: *speed* = *distance*/*time*, or $V = D/T$. Velocity is built out of this—for, velocity is speed in a direction. This, Leibniz will say, is the proper concept (dimension) that a physicist must consider when studying the motion of bodies, not simply speed. But, even here Leibniz sees additional problems for Descartes. For instance, in *Specimen Dynamicum*, he writes:

> Descartes correctly distinguished between velocity and direction, and also saw that the outcome of a collision between bodies is the state which is least different from the prior state. But he did not calculate that least change correctly, because he changed either only the direction or only the velocity, when the change should be determined by both at the same time. But he could not see how this could be done; because he was concerned with modalities rather than realities, he could not see how two such heterogeneous things could be compared and considered at the same time.[25]

By 'velocity' in this passage Leibniz almost surely means 'speed'. Although he recognizes that Descartes took there to be a relation between the speed of a body and the direction in which it moved, because he followed the principle of changing the least number of variables (which in an equation is its parts) in describing collisions, he changed only the speed of the body or its direction. (Note here that this appears to be what Euler would later refer to as the *Principle of Least Possible Action*.) Leibniz says that what Descartes should have done was realize that both needed to vary together. The trouble, he thinks, is Descartes's not being able to see how the two dimensions were (mathematically) related such that he could produce an enumeration of *velocity* (which would be the dimension of speed in a direction).

It is not crystal clear what the remark about paying too much attention to 'modalities', ignoring the 'realities', means, but I suspect that Leibniz is referring to what he takes to be Descartes's rationalistic (or idealist) emphasis on the system of enumeration (the modalities), where the charge seems to be that Descartes failed to check the system he was constructing against observation of the *things* (which are the realities).[26] In other words, Leibniz seems to think, contrary to what Descartes says in the magnet case above, that Descartes put too much trust in a rationalistic construction of physics—where reason and definition played too great a role over empirical observation. This seems to be connected to Leibniz's insight, expressed earlier in the *Specimen Dynamicum*, that observations showed him that the equation *mass* × *velocity* = *force* did not adequately describe falling bodies.[27] This required him to consider new combinations of these simpler enumerations (that is, *mass* and *velocity*). As we now know, he came to think that the equation *mass* × *velocity* squared = *force*, or $m \times v^2 = f$, better captured by way of description his observations of falling bodies.

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[23] Letters of Euler, p. 236.

[24] Letter LXXVIII, 22 November 1760, *Letters of Euler*, pp. 262–5.

[25] Gottfried Leibniz, 'Specimen Dynamicum', in *G. W. Leibniz: Philosophical Texts*, translated by R. S. Woolhouse and Richard Francks (Oxford: Oxford University Press, 1998), p. 160.

In the *Discourse on Metaphysics*, Leibniz is again found worrying about the Cartesian enumeration of dimensions. In Section 17, for instance, we find him concerned about the dimension of *quantity of motion*, and how Descartes seems to have wrongly taken it to be coextensive with the dimension of *force*. The criticism unfolds as follows. Let there be two bodies, *a* and *b*, where *a* has 1 unit of size and *b* 4 units of size (thus *a* will be found in *Z* as a member of the class whose number is **1**, *b* in the class whose number is **4**). So, *b* is 4 times the size of *a*. Now, experience reveals that the force required to lift *a* a distance of 4 units of height will be equal to the force required to lift *b* a distance of 1 unit of height.

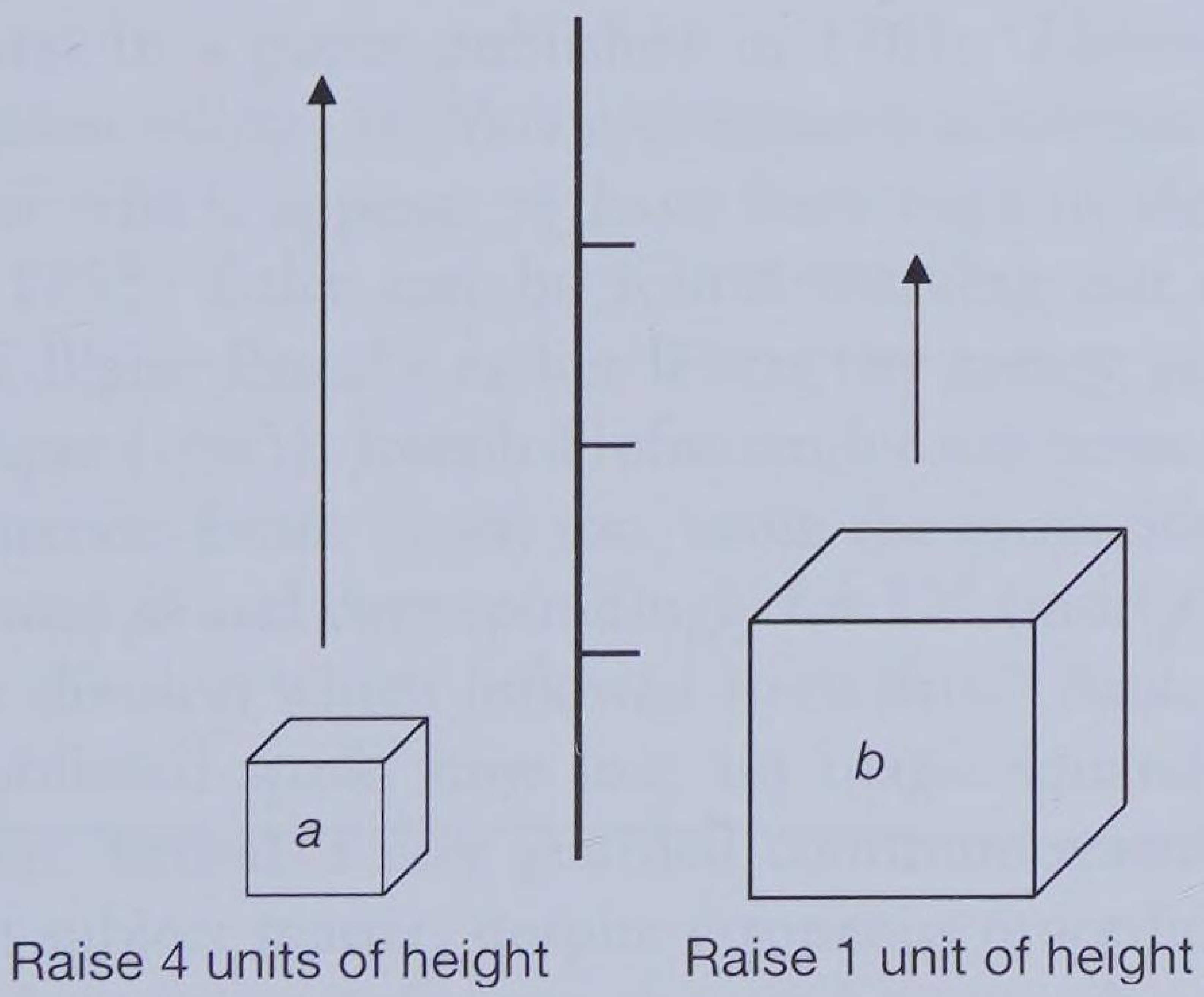

Figure 8.1

In their descent, *a* will fall 4 times the distance of that of *b*, and yet the speed that *a* will acquire on its descent will only be twice that of the speed that *b* will acquire. Using Descartes's equation for quantity of motion, which we know is *size* × *speed* = *quantity of motion*, the calculated quantity of motion of *a* will be half that of *b*. For example, if the speed that *b* acquires on its descent is 2 units of speed, the speed that *a* acquires will be twice that, or 4 units of speed. Thus, the quantity of motion of *a* is 4 units of quantity of motion, whereas that of *b* is 8 units of quantity of motion. This is important, for as *given* in the example, their respective forces are *equal*. And so, their respective quantities of motion differ while their respective forces are equal. Thus, Leibniz concludes, quantity of motion is not coextensive with force.

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[26] In 'Remarks on M. Foucher's Objections' (1695), in Woolhouse and Francks, pp. 184–5, Leibniz refers to *abstract* lines and points as modalities.

[27] Leibniz (Woolhouse and Francks), p. 165.

Synthesis was not only a method for the making of equations. As Leibniz and others of the period were beginning to understand, there was something conceptually about a combinatorial system that was connected to mathematics, and in particular, to the foundations of arithmetic. I want to turn now to looking carefully at this connection.

9

Synthesis and the Principle of Addition

The principle of addition, as it was called by Carnap, accounts for the additive property of units of ratio scales. Of course, the principle accounts for the additivity of *numbers* as well. The principle expresses *the* conditions necessary (and sufficient) for addition—period. Earlier I referred to the concept that this principle expresses as *a group*. The use of the word *group* (*groupe* in French) in this context appears to have originated in the letters of Évariste Galois (1811–1832),[1] although the modern concept would not be formally defined until Arthur Cayley (1821–1895). Niels Henrik Abel (1802–1829) is also counted among the concept's progenitors. Historians of mathematics have noted that prior to Cayley, *group* was mostly associated with the idea of a *permutation group*.

Even so, historians have located the concept at work in the eighteenth century, in the work of Euler, for instance, which was in turn rooted in work of the seventeenth century. In a paper published in 1761, '*Theoremata circa residua ex divisione potestatum relicta*', in *Novi commentarii academiae petropolitanae* 7,[2] an early version of which appears to have been read to the Berlin Academy on 13 February 1755, Euler can be found working out the basics of the group concept. In Blaise Pascal's earlier letters (by ninety years) related to his *Triangle Arithmétique* (1665), Joseph Hofmann locates notes 'probably affected by Roberval's influence. From them, too, came the formation of the system of residues for 10^k (mod p) and correspondingly for 12^k (mod p) for several values of p, and tests the division which followed from this.'[3] And, connected to this was Pascal's unpublished work, now lost, on magic squares.[4] Hofmann goes further to note that 'Leibniz's very guarded communications with Oldenburg concerning related subject matter, despite erroneous procedure contained in it, again anticipated the concepts of group theory.'[5] So, we should find what I am looking for, the group concept, in *some* form, as early as the seventeenth and eighteenth centuries.

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[1] Carl B. Boyer, *A History of Mathematics* (Princeton: Princeton University Press, 1985), pp. 638–9.

[2] Euler, 'Theoremata circa residua ex divisione potestatum relicta', first published in *Novi Commentarii Academiae Pretropolitanae*, 7 (1761), pp. 49–82. This could be found at the time of writing on the excellent website *The Euler Archive*, http://math.dartmout.edu/~euler.

[3] Joseph Ehrenfried Hofmann, *Classical Mathematics* (New York: Barnes and Noble, 2003), originally published by Philosophical Library, Inc., in 1959, p. 28.

[4] Hofmann (2003), p. 28.

[5] Ibid., pp. 59–60.

In order to get a sense of what we will be looking for, we will start by building conceptually up to the concept of a group. Let's begin with what mathematicians call a *semigroup*. A semigroup is a set of objects S, together with a binary operator $\oplus$, where the set is closed with respect to the operator and the operator is associative. Closure is a relatively simple notion to understand. Suppose that S is the set $\{a, b, c\}$, then S is closed with respect to $\oplus$ if in every combination of the members of S—where a combination looks like $a \oplus b$, $a \oplus c$, $b \oplus a$, $c \oplus a$, and so on—the 'sum' of the combination is also a member of S. An ancestor of the closure concept, I believe, is expressed by Leibniz in a mathematical fragment, when he writes, 'Quae similiter determinantur similia sunt.'[6] This can be translated, 'Things similarly determined are similar.' Invoking Euclid's Common Notion 2, from Book I of the *Elements*, Descartes relies on what I believe is a related claim: 'If you add equals to equals the results will be equal.'[7] At the very least, I think, Leibniz's remark is consistent with the claim that when numbers, for example, are added together the result is a number. When adding two numbers, their sum could never be, say, a horse. By way of mathematical operations, such as addition, numbers beget numbers. A clear advance on both Descartes and Leibniz here would be the expression, 'The successor of any number is a number', attributed to Peano.[8]

Associativity is also a simple notion to understand (we all learned it in secondary school). The binary operator $\oplus$ is associative whenever it meets the following condition: for any member of S, the *order* in which one 'adds' the members will yield the same result. So, for instance, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$; where if we started 'adding' inside the parentheses, adding a and b first, and then taking the sum and adding it to c, we would get the very same result were we instead to start by adding b and c, and then taking the sum and adding it to a. Represent this semigroup of the set S and the above binary operator as $(S, \oplus)$.

A *monoid* is a semigroup $(S, \oplus)$ with an element e, which is a member of S, and which serves as the monoid's *identity element*.[9] Such an element can be understood as follows: where e is the identity element, for any member x of S, $x \oplus e = x$; so, $a \oplus e = a$, $b \oplus e = b$, and so on. This is not difficult to grasp. In ordinary addition, for instance, 0 is the identity element. So, $3 + 0 = 3$,

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[6] Gottfried Leibiz, *Leibnizens mathematische Schriften*, C. I. Gerhardt (ed.), 7 vols (Berline, 1846–1860). This is found in vol. 1, p. 544.

[7] AT VIIIA 9; CSM I 197.

[8] See, for example, Jill Vance Buroker's *Kant's Critique of Pure Reason: An Introduction* (Cambridge: Cambridge University Press, 2006), p. 69. Buroker finds support for the association with Peano in Gordon Brittan's *Kant's Theory of Science* (Princeton: Princeton University Press, 1978), p. 48.

[9] Some have called this a 'nullifying' element, indicating that it is able to nullify the operator's power (in this case $\oplus$) to affect members of S. See, for example, Saunders MacLane and Garrett Birkhoff's *Algebra* (New York: MacMillan Company, 1967), p. 61.

456 + 0 = 456, and so on. In ordinary multiplication, 1 is the identity element. So, 3 x 1 = 3. The monoid that we are developing here, then, is the set S, the binary operator $\oplus$, and the identity element e, which can be represented (S, $\oplus$, e).

A *group* is a monoid (S, $\oplus$, e) in which every element of S is 'invertible'. The idea here is that each member of S has an *inverse*, which when 'added' together yields the identity element. So, where we represent '$-x$' as the inverse of 'x', such that $x \oplus -x = e$, then $a \oplus (-a) = e$, $b \oplus (-b) = e$, and so on. Again, this is not all that difficult to grasp. In ordinary addition, −3 is the inverse of 3, since 3 + (−3) = 0; in ordinary multiplication, 1/3 is the inverse of 3, since 3 x 1/3 = 1. Notice that in a group (and a monoid for that matter) the identity element and the binary operator are importantly interrelated, which in turn determines for a group (but not for a monoid) the elements' inverses.

Where S is a set of objects and $\oplus$ is a binary operator, a group meets these four conditions:

1) S is closed with respect to $\oplus$.
2) $\oplus$ is associative.
3) S includes an identity element.
4) S includes inverses for every one of its elements.

An Abelian group (named after Abel, mentioned earlier) would include a fifth condition, namely, the operator would be commutative. A more formal way of putting the above (ignoring the Abelian feature) can go as follows. Where '$x \in$ S' means that x is an element of set S:

1) If $a \in$ S and $b \in$ S, then $a \oplus b = c$, where $c \in$ S.
2) If $a \in$ S, $b \in$ S, and $c \in$ S, then $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
3) There is an element $e \in$ S, such that if $a \in$ S, then $a \oplus e = a$.
4) If $a \in$ S, then there is an element $-a \in$ S, such that $a \oplus (-a) = e$.

The above four conditions express what Carnap called the principle of addition. In meeting *these* conditions, every ratio scale is additive, and is therefore subject to mathematical treatment. Let's pause and consider how the four conditions apply in understanding a system of elements as a mathematical system by considering some actual cases.

Let S be the set {0, 1, 2}, and let the binary operator $\oplus$ be defined as follows: 'a $\oplus$ b' = the *remainder* of a + b divided by 3 (where '+' here is ordinary addition). Some mathematicians, Euler for instance, referred to the remainder here as the *residue*.[10] This looks a bit complicated, but it is not. Take, for example, 1 $\oplus$ 2. We can solve for x in the equation $1 \oplus 2 = x$ by first adding 1 and 2, and then dividing their sum by 3. The solution is the *remainder*. So, the sum of 1 + 2 is 3. We divide 3 by 3. Since 3 goes into 3 exactly 1 time, there is no remainder. The remainder, in other words, is 0. So, $1 \oplus 2 = 0$. Let's try

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[10] The Euler Archive, pp. 49–82.

2 ⊕ 2. The sum of 2 + 2 is 4. We divide this by 3. As 3 goes into 4 only once, and yields a remainder of 1. That is,

$$\begin{array}{rl} 1 & \\ 3\overline{)\,4} & \\ -3 & \\ \hline 1 & \longleftarrow \text{remainder} \end{array}$$

So, $2 \oplus 2 = 1$. We could produce a table of all the possible combinations of the elements in S:

Table 9.1

⊕	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

The above meets the conditions for being a group:

1) S is closed with respect to ⊕. For every application of ⊕ on the elements of S, the result is a member of S.
2) ⊕ is associative. This is easily demonstrated. Consider whether $(1 \oplus 2) \oplus 1 = 1 \oplus (2 \oplus 1)$. Starting with the left side of the equation, inside the parentheses, we know (by now looking at the above table) that $1 \oplus 2 = 0$. Take the sum, 0, and now complete the procedure, which is to add $0 \oplus 1$. The table tells us that $0 \oplus 1 = 1$. Now, work the right side of the equation, again starting with inside the parentheses. The table tells us that $2 \oplus 1 = 0$. Now complete the procedure, which is to add $1 \oplus 0$. The table tells us that $1 \oplus 0 = 1$. Thus, the left and right sides are equal: $1 = 1$. The equation holds for any combination of elements of S.
3) S has an identity element, 0. For example, $1 \oplus 0 = 1$, $2 \oplus 0 = 2$.
4) Every member of S has an inverse: $1 \oplus 2 = 0$, and $2 \oplus 1 = 0$. 2, then, is the inverse of 1, and vice versa. 0 is its own inverse.

Given that the above set S with binary operator ⊕ meets the four conditions, (S, ⊕, 0) is a finite group. It is a genuine mathematical system. Thus, we are now in a position to understand Carmichael's claim, noted back in Part I, which said, 'a set of objects, with the associated rule or rules of combination, is called a *system*, or more explicitly, a *mathematical system*'.[11]

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[11] Robert Carmichael, *Introduction to the Theory of Groups of Finite Order* (New York: Dover Publications, 1956, originally published by Ginn and Company, 1937), p. 4.

The above group is what mathematicians today call 'addition *modulo* 3' or 'mod-3 addition'. Earlier, it was noted that, according to Hofmann, Pascal was able to formulate the idea of a more general modulo p. Given his interest in modulo 12, assuming that Hofmann is right about Pascal's interest,[12] I suspect that Pascal was working on what is today called *clock time*. There are 12 elements, and as any child knows, in clock time (at least one version of it) the addition operation instructs one as follows: if it is 5 p.m. and we add 7 hours, we get 1 a.m., or if it is 10 p.m. and we add 4 hours, we get 2 a.m.; so, $5 \oplus 7 = 1$, $10 \oplus 4 = 2$, and so on. The cycles of clocks and calendars are examples of mathematical systems underwritten by the group concept.

In mod-3 addition, instead of writing the equation $1 \oplus 2 = 0$, we can write it as: $1 + 2 = 0$ (mod-3). For clock time, we could do the same and rewrite $5 \oplus 7 = 1$ as $5 + 7 = 1$ (mod-12). As Hofmann notes, then, when Pascal is looking at modulo p, the idea is that there are p elements in the set, where if a, b, and c are elements, and the sum of a and b is c, mod p arithmetic could be written $a + b = c$ (mod p).

It should be said that we are not limited by the operation of addition. We can, for instance, develop modular multiplication. For example, let S be the set {0, 1, 2, 3, 4}, and let our binary operator be $\otimes$. Let '$a \otimes b$' = the remainder of the product of a x b divided by 5. Thus, $3 \otimes 3 = x$ is to be solved by first multiplying 3 by 3, which is 9, and dividing this product by 5. Since 5 goes into 9 only once, there is a remainder of 4. So, $3 \otimes 3 = 4$. The table for S and $\otimes$ is:

Table 9.2

⊗	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

The above meets the conditions of a group. It is mod-5 multiplication. Notice that 1 is the identity element. 0 gets special treatment by mathematicians when it comes to multiplication (and subsequently division), though we need not go into this here.

We could find sets of objects that lend themselves to being operated on by *two* distinct binary operators—for example, mod-5 addition and mod-5 multiplication. The initial four conditions (expressed earlier as postulates or rules) of a group will need to be modified and expanded. Putting two distinct operators in play, where each by itself is an operator of a group, describes a *new* mathematical structure called a *field*. The five conditions for this are:

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[12] The trouble is that Hofmann does not tell us in what letter or letters the material is located.

1) The given set of elements must be closed with respect to the two operations.
2) The set must contain two identity elements, one with respect to each of the two operations.
3) For each in the set, there must also be in the set an inverse element with respect to the first operation (addition).
4) For each element in the set, except for the addition identity, there must also be in the set an inverse element with respect to the second operation (multiplication).
5) Both operations must have the associative and commutative property, and the second operation (multiplication) must be distributed with respect to the first (addition).[13]

This structure, a field, is important for a mathematical physics, since it is what underwrites a vector space, which was used by Leibniz, Newton, and others to deal mathematically with *motion* (and later *force*), here regarded as a vector (an arrow whose magnitude in length represents its speed). Dealing mathematically with direction and speed together like this, recall, was something that Leibniz says Descartes seemed unable to do.[14]

Although it is not clear whether Leibniz fully understood what this underlying field structure was, in his mathematical writings he demonstrates a sophisticated grasp of related structures—structures that turn out to be building blocks of a field. The best way to see what Leibniz understood, I think, is to start with a more contemporary expression of his insight and then to work back. I will start with Whitehead and Russell's *Principia Mathematica*.

In the section dealing with proposition *22, titled 'Calculus of Classes', Whitehead and Russell lay out the postulates underwriting what they refer to as the *algebra of logic*. Let me cite portions of the discussion that follows directly on proposition *22.05:

> We assume a class K, with two rules of combination, namely $\cup$ and $\cap$; and we then require the following ten postulates:
>
> Ia. $a \cup b$ is in the class whenever a and b are in the class.
> Ib. $a \cap b$ is in the class whenever a and b are in the class.

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[13] This is taken word for word from John Yarnelle's *Finite Mathematical Structures* (Boston: D. C. Heath and Co., 1964), p. 6.

[14] Here, Leibniz is well aware of Descartes's *Optics*, which does treat light in terms of vectors. What Leibniz seems to have in mind, when criticizing Descartes, is Descartes's not understanding the mathematics involved that *underwrite* the vector concept.

IIa. There is an element Λ such that $a \cup \Lambda = a$ for every element a.
IIb. There is an element V such that $a \cap V = a$ for every element a.[15]

Postulates IIIa and IIIb define commutability for both operators, and IVa and IVb define distribution for the operators. The next postulate is:

V. If the elements Λ and V in postulates IIa and IIb exist and are unique, then for every element a there is an element $-a$ such that

$a \cup -a = V$ and $a \cap -a = \Lambda$. [16]

The above looks suspiciously as if it covers the same conceptual ground covered by the earlier four postulates that defined a group. And, given that two operators are in play, with what appear to be two identity elements (one for each operator), where commutability and distribution holds, it would appear that the above defines a field. But, it defines neither a group nor a field.[17] Let me explain.

To be sure, postulates Ia and Ib clearly express closure. IIa appears to express the condition for an identity element, namely Λ, for the operator $\cup$; IIb the condition for an identity element, V, for the operator $\cap$. Now, Λ and V are defined by Whitehead and Russell as the *null-class* and the *universal-class* respectively.[18] The former is the class containing nothing; the latter the class containing everything. And, the operators $\cup$ and $\cap$ are *union* and *intersection* respectively. Thus, IIa is saying that for every element a in the class, where a may be a class, if we union a with the *null-class*, that is if $a \cup \Lambda$, the new class will be a (since the null-class contains nothing, when we bring the classes together, nothing is added to a). IIb is saying that for every element a in the class, if we intersection a with the *universal-class*, that is if $a \cap V$, the new class will simply be a (for the only thing that a and the *universal-class* have in common is a). So far, so good.

The difference between defining a field and something else (I am about to say what the latter is) begins to come to light when we consider the fifth postulate. Here, it would appear that the postulate is defining inverses. But, it is not. Consider again IIa, where $a \cup \Lambda = a$. The inverse of a, namely $-a$, must be determined with respect to the operator $\cup$ and identity element Λ, such that $a \cup -a = \Lambda$. But, notice that the fifth postulate tells us that $a \cup -a = V$ (and ignores the proper identity element Λ). That is, the 'inverse' here is being defined in terms of an operator and identity element that are not related in the initial postulates. We are able to get the identity element Λ by way of $a \cap -a$.

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[15] Alfred North Whitehead and Bertrand Russell, *Principia Mathematica (to *56)* (Cambridge: Cambridge University Press, 1990) first published 1910, p. 205.

[16] Whitehead and Russell (1990), p. 206.

[17] In a colloquium talk I gave to the Bloomsburg mathematics department, Drue Coles brought it to my attention during the discussion period that this structure was a Boolean algebra. I am indebted to him and to the other mathematicians who attended for the very helpful discussion.

[18] Whitehead and Russell (1990), p. 206.

But, again, $\cap$ is not the operator to which the identity element Λ was originally related. Recall what was earlier noted about the relation between the operator, identity element, and inverses—they are internally (logically) related. To help understand what the problem is, consider the earlier cases of ordinary addition and multiplication. For ordinary addition, where + is the operator and 0 the identity element, -3 was said to be the inverse of 3, since $3 + (-3) = 0$. For ordinary multiplication, where x is the operator and 1 the identity element, 1/3 was said to be the inverse of 3, since $3 \text{ x } 1/3 = 1$. Understood in terms of ordinary addition (operator +, identity element 0) and multiplication (operator x, identity element 1), Whitehead and Russell's fifth postulate in effect seems to be telling us something like the following: for the operator + there must be some inverse of 3, call it x, that yields 1, that is $3 + x = 1$, and for the operator x there must be some inverse of 3, call it y, that yields 0, that is $3 \text{ x } y = 0$. But, for ordinary addition 1 is not the operator's identity element, and so 1/3 is not 3's *inverse*, and for ordinary multiplication 0 is not the operator's identity element, and so 0 is not 3's *inverse*. This is enough to show that Whitehead and Russell are not dealing with the concept of *inverse* when introducing $-a$ in the fifth postulate.

The problem is solved if we take a and $-a$ to be classes, where the two are what are called *complements* of one another. Class a contains everything that class $-a$ does not, and vice versa. So, if a is the class of rational animals, $-a$ is the class of everything that is not a rational animal. This will include frogs, rocks, stars, paper clips, the number 17, and so on. The classes are not, strictly speaking, *opposites*. Strictly speaking, there is nothing that is opposite of a rational animal. A flea is as much an *opposite* of a rational animal in the sense of complement, I suspect, as is the Pythagorean Theorem. So, when we union a with $-a$, that is $a \cup -a$, we get the new class which includes a and everything other than a. This new class is the class of everything, or the class V—the *universal-class*. And, when we intersection a with $-a$, that is $a \cap -a$, we get the new class which includes everything that a and $-a$ have in common, which is nothing at all. So, this new class is the class Λ—the *null-class*. This is exactly what the fifth postulate claims. There are no proper inverses in this system. And so, at best, the structure that Whitehead and Russell are describing with the above postulates, the two binary operators, and the two identity elements (assuming they are such), is a structure of two monoids.

Recall that above we introduced the idea of a structure with two groups. This is a field. If we 'demoted' one of the groups to monoid status (the monoid would meet the conditions of closure, associativity, and identity element—but it would not have inverses. We demote it by redefining it or changing some of the elements in the set over which it operates), where the result was a structure with one group and one monoid, we would have a *ring*. If we went further and 'demoted' the other group to monoid status, where we now had a structure with two monoids, this structure would be what is called a *Boolean algebra*. Given this is the structure that Whitehead and Russell describe, the mathematical structure of the *Principia Mathematica* is at best a Boolean algebra. It is not a field. If having the structure of a Boolean algebra can account for a system's being *mathematical*, then the *Prinicipia Mathematica* describes an inherently mathematical system. That having been said, its structure is not a group (or, since two operations are being considered, a field), and so if the ground floor of a mathematical system requires that at least one of the operators (and the set of objects over which the operator operates) is a constituent of a group, the *Principia Mathematica* is not a mathematical system. The same, then, would apply to an assessment to Leibniz's logical calculi.

Although Whitehead and Russell mention several who had worked, or were working, on the foundations of arithmetic, the particular formulation of their crucial postulates appears to be based on Edward Huntington's 1904 paper, 'Sets of Independent Postulates for the Algebra of Logic', published in *Transactions of the American Mathematical Society*.[19] Huntington, a Harvard mathematician, had published a series of papers that led to the paper on the algebra of logic, the first three in 1903—one working out the group concept, the second working out the field concept, and the third working out the concept of real quantities.[20] But it is the 1904 paper on the algebra of logic on which Whitehead and Russell focus.

In the 1904 paper, Huntington adopts the binary operator symbol $\oplus$, noting that it was 'used by Leibniz for the same purpose about 1700'.[21] Here is an important (although easily missed) connection between the mathematical structure described in the *Principia Mathematica* and certain insights that Leibniz had about that structure. The symbol appears in a logico-mathematical fragment, in the context of a definition. Leibniz writes:

> B $\oplus$ N ∞ L significant B esse in L, seu L continere B; at B et N simul constituere vel componere L.[22]

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[19] Edward V. Huntington, 'Sets of Independent Postulates for the Algebra of Logic', *Transactions of the American Mathematical Society*, vol. 5, no. 3 (July 1904), pp. 288–309. Historians of mathematics typically locate the origin of the group concept (or, at least the first formalization of it) in Peano's *The Principles of Arithmetic, Presented By A New Method* (1889). But, a close study of the text, I think, shows that Peano may not in the end have the concept of an inverse. Arguably, like Russell and Whitehead, Peano seems at best to articulate a Boolean algebra. Huntington, I contend, is among the first to *clearly* articulate the four conditions of a group structure, especially making clear the concept of the inverse. For my examination of inverses, I am using Hubert Kennedy's translation of Peano's work, in *Selected Works of Giuseppe Peano* (Toronto: University of Toronto Press, 1973), pp. 109–112.

[20] Edward V. Huntington, 'Two Definitions of an Abelian Group by Sets of Independent Postulates', *Transactions of the American Mathematical Society*, vol. 4, no. 1 (January 1903), pp. 27–30; 'Definitions of a Field by Sets of Independent Postulates', *Transactions of the American Mathematical Society*, vol. 4, no. 1 (January 1903), pp, 31–7; 'Complete Sets of Postulates for the Theory of Real Quantities,' *Transactions of the American Mathematical Society*, vol. 4, no. 3 (July 1903), pp. 358–70.

[21] Huntington (1904), p. 292.

[22] Gottfried Leibniz, *De Philosophischen Schriften von Gottfried Wilhelm Leibniz*, Band VII, von C. I. Gerhardt (New York: George Olms Verlag, 1978), p. 237.

Here, Leibniz tells us that '$B \oplus N \infty L$' signifies that B is in L (or that L contains B), and that B and N together constitute or compose L.[23] Clearly, B, N, and L are classes, and the binary operator $\oplus$ turns out to be analogous to $\cup$, or *union*—though when Leibniz was flirting with an intensionalist interpretation of logic, $B \oplus N$ could be read as the intersection of B and N.[24] Be that as it may, here Leibniz is saying $B \cup N = L$.

That Leibniz is dealing with sets and not with quantities is clear from an earlier fragment, which deals with a similar expression:

> if A and B taken together coincide with L, then A, or B, will be called the *inexistent* (*inexistens*) or the *contained*; and L will be called the *container*. However, it can happen that the container and the contained coincide, as for example, if (A and B) $= L$, and A and L coincide, for in that case B will contain nothing which is different from A, $A + B = L$ signifies that A is in or is contained in L.
>
> *Scholium.* Although A and B may have something in common, so that the two taken together are greater than L itself, nevertheless what we have here stated, or now state, will still hold. It will be well to make this clear by an example: Let L denote the straight line RX, and A denote a part of it, say the line RS, and B denote another part, say the line XY. Let either of these parts, RS or XY, be greater than half the whole line, RX; then certainly it cannot be said that $A + B$ *equals* L, or $RS + XY$ *equals* RX. For inasmuch as YS is a common part of RS and XY, $RS + XY$ will be equal to $RX + SY$. And yet it can truly be said that the lines RS and XY together *coincide* with the line RS.[25]

Understood as quantitative magnitudes, $RS + XY$ will be greater than RX (in fact, the sum will be greater by SY). Leibniz notes that the 'overlap' of RS and XY will not count twice in the system that he is developing. The overlap, we might say, is redundant. His insight seems to be that every point in RS and in XY will be in RX, and vice versa—in other words, the points that $RS + XY$ contain *coincides* with the points RX contains. In terms of the overlap, those points that RS and XY have in common (contained in SY) do not *increase* the new class $RS \cup XY$ by a magnitude of SY. So, Leibniz clearly understands the difference between treating RS, XY, and so on, as quantities and treating them as sets. And, in his logical calculi, he treats them as sets.

According to C. I. Lewis, George Boole, whose mathematical logic looks to be in part rooted in Leibniz's, would eliminate the concern about the overlap. Where '$x + y$' symbolizes the class formed by combining the two distinct classes,

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[23] My translation of this matches that of C. I. Lewis. He translated two fragments of Leibniz's mathematical work, appending them in his *A Survey of Symbolic Logic* (Berkeley: University of California Press, 1918), pp. 291–305. This particular passage is found on page 298.

[24] Lewis (1918), p. 291.

[25] This comes from Gehrhardt, *Die Philosophischen Schriften von G. W. Leibniz*, Band VII, '*Scientia Generalis. Characteristica*,' XIX. I cite Lewis's translation, Lewis (1918), p. 292.

x and *y*, 'it is a distinctive feature of Boole's algebra', Lewis notes, 'that *x* and *y* in $x + y$ must have no common members'.[26] According to Lewis, Boole reads the relation x + y as 'that which is either *x* or *y but not both*'.[27] Lewis refers to this as a *complete* disjunction.[28]

Nicholas Rescher also works out some important details of the evolution of Leibniz's logical calculi that are relevant to the present discussion.[29] As Rescher understands it, in its second stage of development (beginning around 1685), found in *Generales inquisitiones de analysi notionum et veritatum*, *Principia calculi rationalis*, *Difficultates logicae*, and (1690) *Primaria calculi logici fundamenta* and *Fundamenta calculi logici*, Leibniz augments his earlier logical calculus with the following sorts of assertions:

(i) If A is proper: A est B iff AB = A,
(ii) If A non est non-Ens, then A est Ens,
(iii) A est non-Ens iff A non est Ens.[30]

Rescher offers one interpretation of how to read the above:

> Let the 'terms' be sets, 'non' complementation, let juxtaposition represent intersection, *Ens* the universal class, and propriety non-nullity. Finally, let 'est' represent inclusion among non-null sets, in accord with the rule: 'A est B' is '$A = \Lambda$' or '$A \subset B$ & $AB \neq \Lambda$' according as B is empty (null) or not.[31]

Although in the first stages of Leinbiz's system 'A est B' would sometimes be cast in intensional terms,[32] by this second stage Rescher notes that the phrasing expresses the extensional concept 'A is contained in B', which the universal proposition 'All As are Bs' expresses.[33] Thus, the above rule could be understood as saying: 'A is contained in B' tells us either A is the null-class or that A is a subset of B and the intersection of A and B is not the null-class. Assertion (i) listed above, then, could be read: If A is proper (that is, A is a non null-class), then A is contained in B if, and only if, the intersection of A and B is A. Recall postulate IIb from *Principia Mathematica*: There is an element V such that $a \cap V = a$ for every element *a*. If B were the universal-class, then clearly A is contained in B. If we intersection A with B, since the only thing both would have in common is A, the result would simply be A. Interpreted this way, Leibniz's (i) looks remarkably like the *Principia Mathematica* postulate. It should be stressed that we can, as we did here, find versions of the postulates of the *Principia Mathematica* in Leibniz's earlier logical calculi.

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[26] Ibid., p. 52. [27] Ibid., pp. 52–3. [28] Ibid., p. 53.

[29] Nicholas Rescher, 'Leibniz's Interpretation of His Logical Calculi', *The Journal of Symbolic Logic*, vol. 19, no. 1 (March 1954), pp. 1–13.

[30] Rescher (1954) locates the second stage in the works listed (p. 7). The assertions are Rescher's rendering (e.g., the use of 'iff' is Rescher's, not Leibniz's).

[31] Ibid., p. 8. [32] Ibid., p. 6. [33] Ibid.

In an earlier stage of Leibniz's logical calculus, *Specimen Calculi universalis* (1679), he makes the following assertions:

1) *a* est *a*. *Animal est animal*.
2) *ab* est *a*. *Animal ratitionale est animal*.
3) *a* non est non-*a*. *Animal non est non animal*.[34]

As in the second stage of development 'est' can be read as 'is contained in'. So, '*a* est *a*' can be read '*a* is contained in *a*'. Given that juxtaposition is intersection, (2) can be read as 'the intersection of *a* and *b* is contained in *a*'. The interpretation that Leibniz himself gives (to the right) is '(the class) rational animal is contained in (the class) animal'. Notice that this differs slightly from the second stage, where we noted assertion (i) says: If A is proper (that is, A is a non null-class), then A is contained in B if, and only if, the intersection of A and B is A. Here, we did not say that the intersection of A and B is *contained in* A. But, if '*a* est *a*' is to be read '*a* is contained in *a*', then we *could* work from 'A and B is A' to 'A and B is contained in A'. So, although slightly different in interpretation, the two stages seem compatible on this point. What is more, (3) can be read thus: '*a* is not contained in non-*a*', where his interpretation of this is: '(the class) animal is not contained in (the class) non-animal'. Here, it seems to me, Leibniz has captured the notion of complements, where *a* and non-*a* (or $-a$) are complements. So, as in the *Principia Mathematica*, $a \cup -a = \mathrm{V}$ and $a \cap -a = \Lambda$.

Relatively early in the evolution of the more *formalized* conceptualization of the group concept (this happens after Leibniz) inverses appear in terms of operations, not in terms of elements. Lewis locates this in Boole's logical work. Lewis says:

> But making the relation $x + y$ completely disjunctive has one advantage—it makes possible the inverse relation of 'subtraction'. The 'separation of a part, x, from a whole, y', is represented by y—x. If $x + z = y$, then since x and z have nothing in common, y—$x = z$ and y—$z = x$. Hence [+] and [–] are strict inverses.[35]

Even so, without the *element* inverses (as opposed to operator inverses), the structure of Boole's mathematical system (not coincidentally) is a Boolean algebra and not a field. What I am suggesting, based on the sort of thing shown above, is that, although Boole will be credited with the *formalization* of the concept, Leibniz certainly appears to have had a grasp of it, and to some extent a rather sophisticated grasp.

What is more, I believe that it can be shown that Leibniz had some insight into structures more complex than the Boolean algebra. How much more exactly is difficult to say, of course. But, his description of the cosmos in his *Monadology* (though some of this is traceable back to the earlier *Discourse on Metaphysics*) suggests that he was flirting with such structures, specifically with either a ring or a field. Allowing, then, for Leibniz's sophisticated grasp of certain mathematical concepts (some still to be discussed), I want to turn now to showing how we can read him as describing (*sans* mathematical jargon) in his *Monadology*, for example, a mathematical structure, setting some of this discussion up with a brief consideration of what he says in the *Discourse*. I will suggest that one of the things he seems to be doing in such texts is describing an ideal-world structure (i.e., a mathematical structure) that underwrites the very possibility of the phenomenal world.

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[34] Gottfried Leibniz, 'Specimen Calculi universalis', *Die Philosophischen Schriften von Gottfried Wilhelm Leibniz*, vol. 7, von C. I. Gerhardt (New York: George Olms Verlag, 1978), p. 224.

[35] Lewis (1918), p. 53.

10

Metaphysics, Mathematics, and Metaphor

Commentators have argued that very early in Leibniz's philosophical career he took mind (or soul) to be a completely active principle.[1] Since it is completely active, it requires some medium *through which* it is able to act. Matter, a passive principle, serves this metaphysical purpose.[2] In acting through matter, mind and matter are *unified*, the resulting unity being a *body* or *material substance*.[3]

Leibniz's motivation for developing this view appears to have been rooted in his concern over the origin of motion. Unlike the Cartesians who had argued that modes, such as size, shape, and motion, could be demonstrated to be necessarily connected to the attribute extension (since the idea of extension is the underlying condition making intelligible the ideas of size, shape, and motion), Leibniz argued that, although this held for size and shape, it simply did not hold for motion. *Shape entails extension*, we might put it, since one could not think of a shaped thing without thinking of an extended thing. And, this holds the other way around—at least if we limit the extension. That is, *finite extendedness entails shape*, since one could not think of a finite hunk of extension without thinking that it has some shape or other. Christia Mercer points out that Leibniz held that one *could* think of finite extendedness without thinking that this hunk of extension *moves*.[4] And so, whereas a mode such as shape exhibits the conceptual (logical) relationship to extension, motion does not. Setting aside the later Cartesians, Descartes himself certainly seemed to be aware of the problem, for even he requires God to contribute a quantity of motion at the creation of the cosmos, which suggests that creating extension in itself was not enough to create motion.

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[1] Christia Mercer and R. C. Sleigh, Jr., 'Metaphysics: The early period to the *Discourse on Metaphysics*', *The Cambridge Companion to Leibniz*, edited by Nicholas Jolley (Cambridge: Cambridge University Press, 1995), p. 88. Mercer writes that part of the contribution in which the position to which I refer is found. She extends an in-depth exploration of this view in her book, *Leibniz's Metaphysics* (Cambridge: Cambridge University Press, 2001). An excellent article that sorts out the views found in the secondary literature on Leibniz (published over the past twenty years or so) is Paul Lodge's 'Garber's Interpretations of Leibniz on Corporeal Substance in the "Middle Years"', *The Leibniz Review*, vol. 15 (December 2005), pp. 1–23.

[2] Mercer and Sleigh (1995). [3] Ibid.

[4] It might appear that, given the shape of a thing is determined by motions, that in thinking of a shaped thing we are thinking of a moving thing. Here, the motion under consideration is not the motion or motions necessary for determining a thing's shape, but the motion of the shaped thing relative to other shaped things.

Leibniz claims that substances include in their natures the origin or principle of their motion. Since extension or corporeality, which the Cartesians took to be the nature or essence of body, is not able to conceptually (and hence metaphysically) account for motion, it follows that extension is not *substantial*.[5] By default, the origin or principle of motion must be something other than extension or corporeality. It must be *in*corporeal. If bodies are material substances, their natures must include an origin or principle of motion. And so, if bodies are material *substances*, their nature must include an incorporeal element. This element is the soul. As I noted above, commentators have shown that relatively early in Leibniz's career, he held that a body or material substance emerges as the result of a soul (or mind) acting through the metaphysical medium of matter. A *material substance*, in other words, is a mind–matter unity. Along these lines, in the *Discourse*, Leibniz offers as an example of an individual substance the particular human being, Alexander the Great. I will consider this example shortly.

Some scholars find a different view developed in Leibniz's later work.[6] For example, he opens the *Monadology* (1714) with:

> The *monad*, of which we will be speaking here, is nothing but a simple substance, which enters into composites; *simple*, meaning without parts.[7]

Insofar as there are no parts, 'neither extension, nor shape, nor divisibility is possible'.[8] Monads are 'the elements of things'.[9] Now, although every monad is distinct,[10] the distinction is not quantitative.[11] Their differences (and hence the plurality of monads) are to be cashed out in terms of a plurality of states or relationships.[12] This, as should become clear in what follows, looks to be connected to what Aquinas says when speaking of the plurality of forms (recall, the multitude they constitute is not quantitative), which results from a 'division' of the immaterial substrate.

By this time in his career, Leibniz was emphasizing the argument that substances are incorruptible, which entails that they are indivisible.[13] Since matter or extension is divisible by its very nature (infinitely so), it follows that it is not substantial. This is different, of course, from the earlier argument that matter or extension was not substantial since it cannot account (conceptually) for the origin of motion. What is more, it is not obvious how we are to square the new view of *monad* as individual substance with the earlier view of *Alexander the Great* as individual substance. There is the inclusion in the later work of the idea of *dominant* monads, which may serve to bridge the conceptual gap, but I think that enough will be introduced in what follows to at least sketch out what I think is a solution, without having to entertain a host of commentators who have dealt with the difficult concept of dominant monad.

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[5] Here, I am relying primarily on Mercer's account, Mercer (2001).

[6] See, R. M. Adams, Leibniz: *Determinist, Theist, Idealist* (Oxford: Oxford University Press, 1994); Daniel Garber, 'Leibniz and the Foundations of Physics: The Middle Years', in *The Natural Philosophy of Leibniz*, K. Okruhlik and J. R. Brown (eds) (Dordrecht: Reidel, 1985), pp. 27–130; Donald Rutherford, *Leibniz and the Rational Order of Nature* (Cambridge: Cambridge University Press, 1995); and Lodge (2005).

[7] Gottfried Leibniz, *Monadology*, in *G. W. Leibniz: Philosophical Texts*, translated and edited by R. S. Woolhouse and Richard Francks (Oxford: Oxford University Press, 1998), p. 268, proposition 1.

[8] *Monadology*, proposition 3. [9] Ibid. [10] Ibid., proposition 9.

[11] Ibid., proposition 8. [12] Ibid., proposition 13.

[13] Letter to Arnauld, draft 28 November/8 December 1686, in Woolhouse and Francks, p. 116.

My aim here is not to side with any particular view worked out in the secondary literature (though several will be relied on in what follows), nor to suggest a view that has not already been suggested by others, but to show that *regardless* of what counts as an individual substance for Leibniz, the world that he describes, in which those substances are found, does more than suggest the idea of a mathematical structure. To show this, I will briefly consider both the earlier and later views.

In the *Discourse*, Leibniz introduces what he calls a substance's *notion*.[14] As I understand it, the notion of x is something like a function, such that if one understands the notion of x one will have a complete understanding of the substance, namely x, that this notion determines. Leibniz's idea goes something like this. Suppose that we are given a domain of predicates, the notion of x will order certain predicates taken from the domain such that the result will express a unique (that is, individual) substance. Thus, if one understood the notion of Alexander the Great, Leibniz says, one would see 'the foundation and the reason for all the predicates which can truly be said to belong to [him]'.[15] What is more, 'if we consider the interconnectedness of things,' he says, 'we can say that in the soul of Alexander there are for all time remnants of everything that has happened to him, and marks of everything that will happen to him—and even traces of everything that happens in the universe'.[16]

So, suppose that we have a domain of predicates that includes p_1, p_2, p_3, and so on, where the notion of Alexander orders the predicates thus: $\{p_{23}, p_{14}, p_{965}, \ldots, p_{22}\}$, where p_{22} is, say, death by poisoning. This unique ordering of predicates expresses Alexander as an individual substance. If one understood Alexander's notion, one could easily deduce, for example, that he died of poisoning.

This, in principle, is no different from understanding the notion (or function) that determines a line.[17] In Section 6 of the *Discourse*, Leibniz says that even if someone were to haphazardly put on paper a number of points, 'it is possible to find a geometrical line whose notion is constant and uniform according to a certain rule, such that the line passes through all the points, and in the same order as they were drawn'.[18] In fact, no matter how random in configuration a line first appears, 'it would be possible to find a notion or rule or equation common to all the points on this line in virtue of which these changes [in the line's configuration] would occur'.[19] It is this sort of passage that prompts my suggesting that *notion* is working along the lines of *function* or *equation*.

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[14] Gottfried Leibniz, *Discourse on Metaphysics*, in *G. W. Leibniz: Philosophical Texts*, translated and edited by R. S. Woolhouse and Richard Francks (Oxford: Oxford University Press, 1998), p. 60. This is Section 8.

[15] Ibid. [16] Ibid.

[17] C. D. Broad argues that there are important differences between notions of species, which include notions of such things as circles, and notions of individuals. 'Leibniz's Predicate-in-Notion Principle', in *Leibniz*, edited by Harry G. Frankfurt (New York: Anchor Books, 1972), pp. 1–18.

As Leibniz himself recognizes it, the equation defining a linear function takes the form $Ax + By + C = 0$. The following describes a line: $2x - 3y = 6$. Here, $A = 2$, $B = -3$, and if we moved 6 over to the left side of the equation, we would see that $C = -6$. Using a coordinate system we could graph this line. This is easily done by determining the x and y intercepts (where the line crosses the x and y axes). If we let $y = 0$ in the equation, we can find the x-intercept, which turns out to be 3. And, if we let $x = 0$ in the equation, we can find the y-intercept, which turns out to be -2. So, the line looks thus:

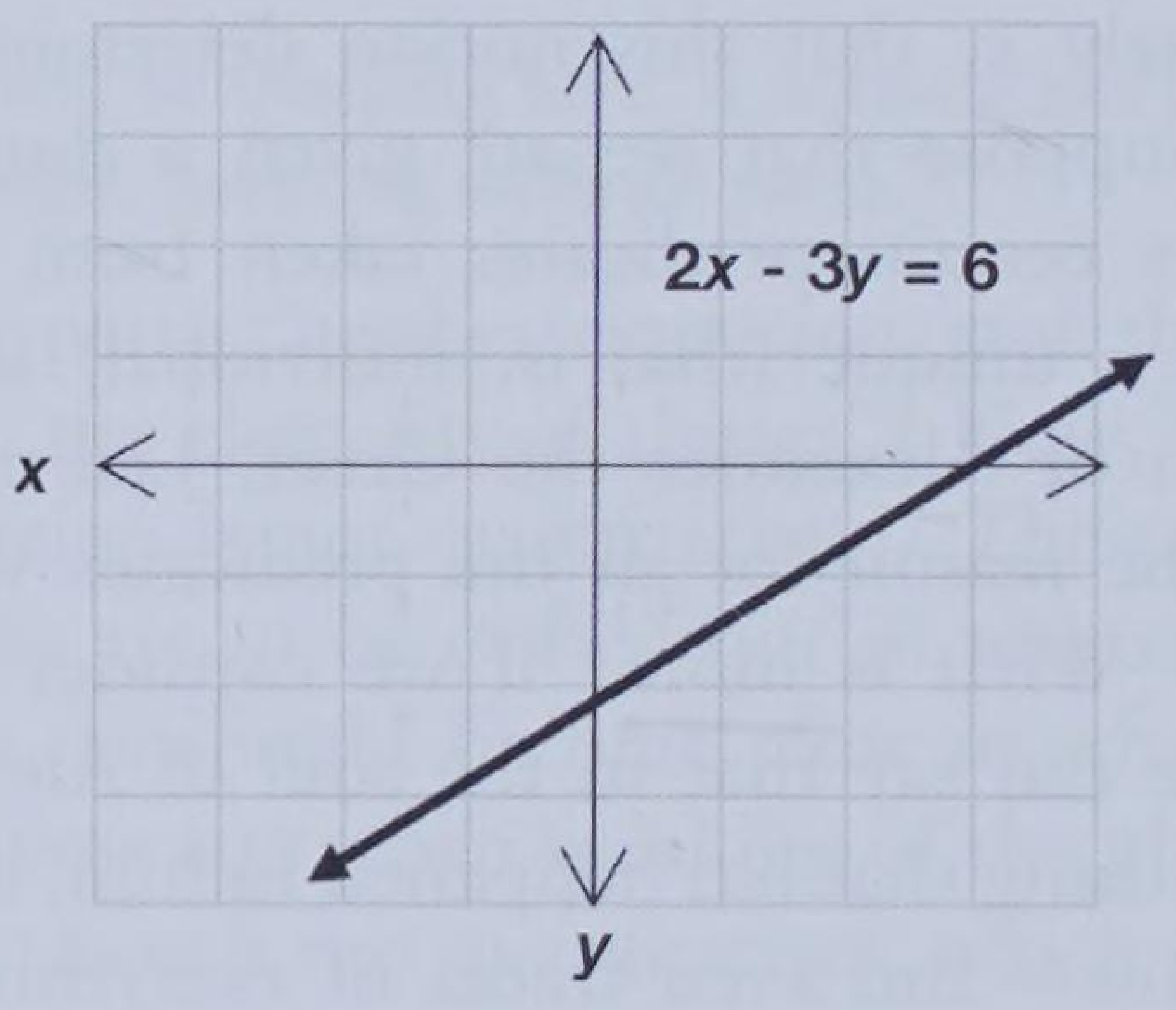

Figure 10.1

The equation here is able to take any x coordinate (of which there is an infinite number) we might consider and pair it with a unique y coordinate (of which there is an infinite number) such that the ordered pairs produced would place us on this, and only this, line. Thinking of the line here as the analogue to Alexander the Great, like the notion (function) determining this unique line, the notion of Alexander determines a unique substance in the cosmos.

To further illustrate what I take to be going on with Leibniz's example of Alexander the Great, let the following be a primitive system of pixels (a very simple computer screen).

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[18] *Discourse*, Section 8, (Woolhouse and Francks), p. 58.

[19] Ibid.

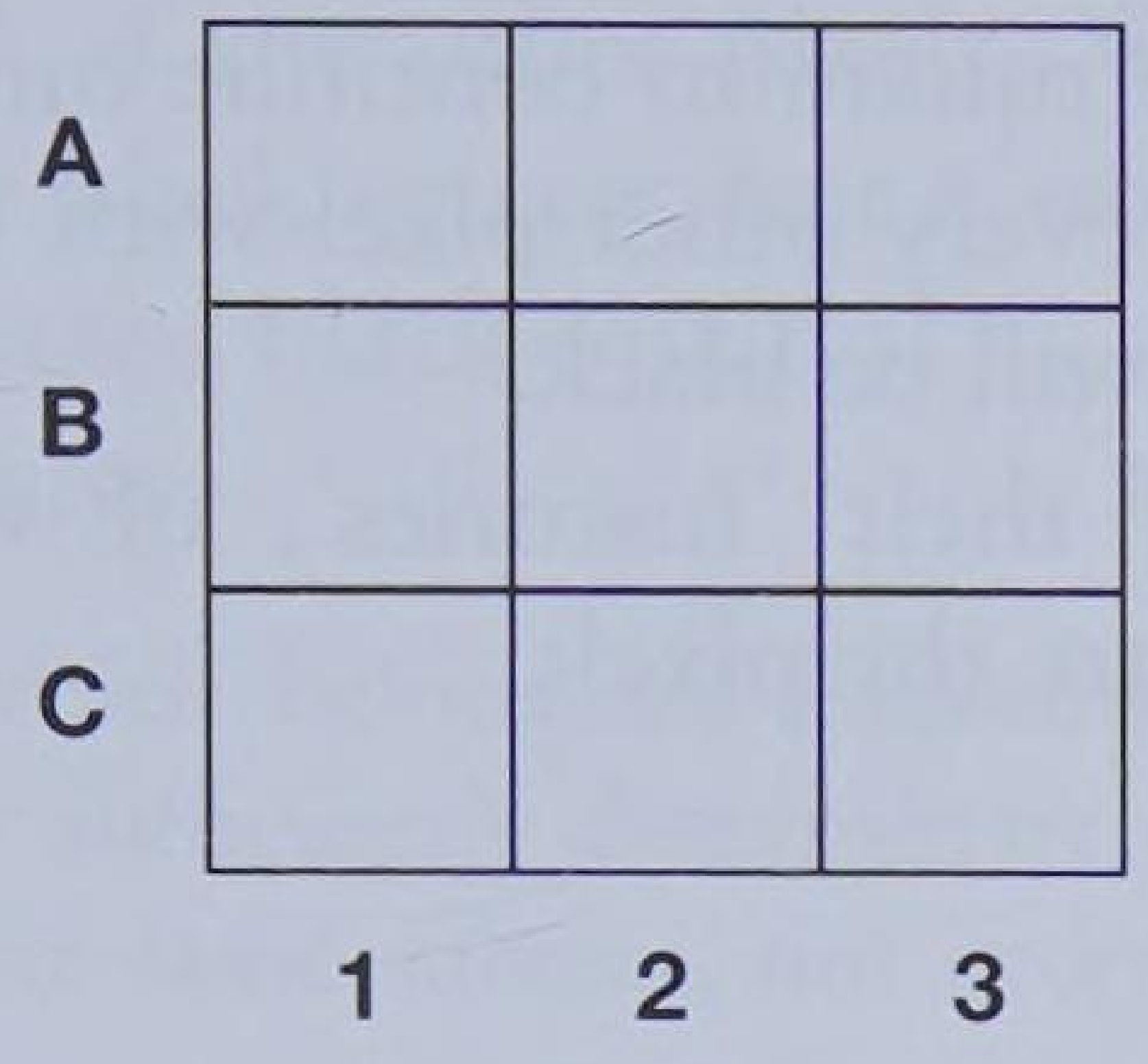

Figure 10.2

A1, B1, and so on, are pixels. Let's now produce the dynamic image of a moving square, where *the moving square* is the analogue to Alexander the Great. What we will have to do is program each pixel so that the entire system, as time unfolds, will work to produce our moving square.

So, let each program entry represent the state of the system at a given time t. Here is the entry for t_1: A1 = white; A2 = white; A3 = white; B1 = white; B2 = white; B3 = white; C1 = black; C2 = white; C3 = white. And so, at t_1 the screen appears thus:

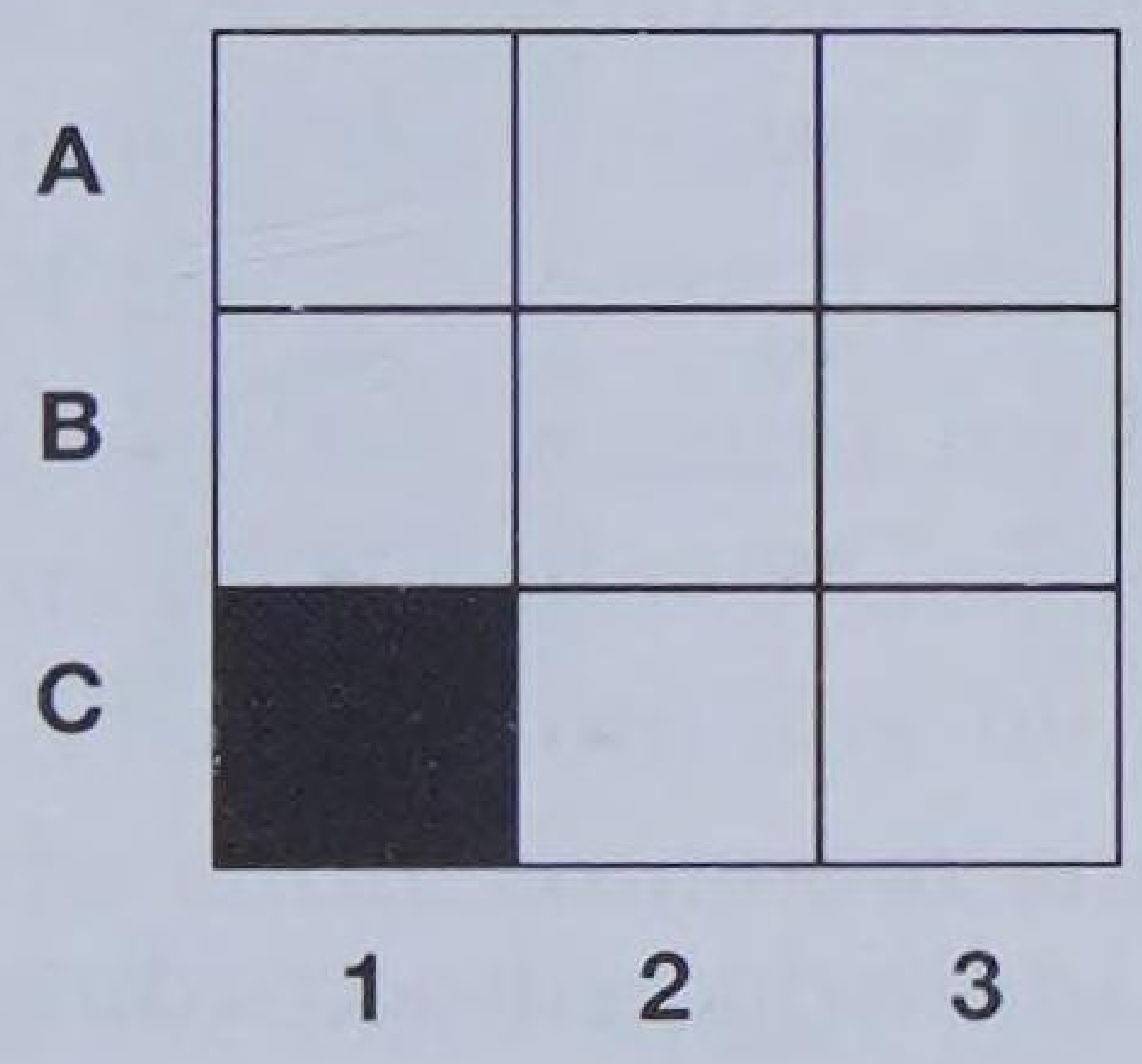

Figure 10.3

Here is the entry at t_2: A1 = white; A2 = white; A3 = white; B1 = white; B2 = black; B3 = white; C1 = white; C2 = white; C3 = white. And so, the screen at t_2 looks thus:

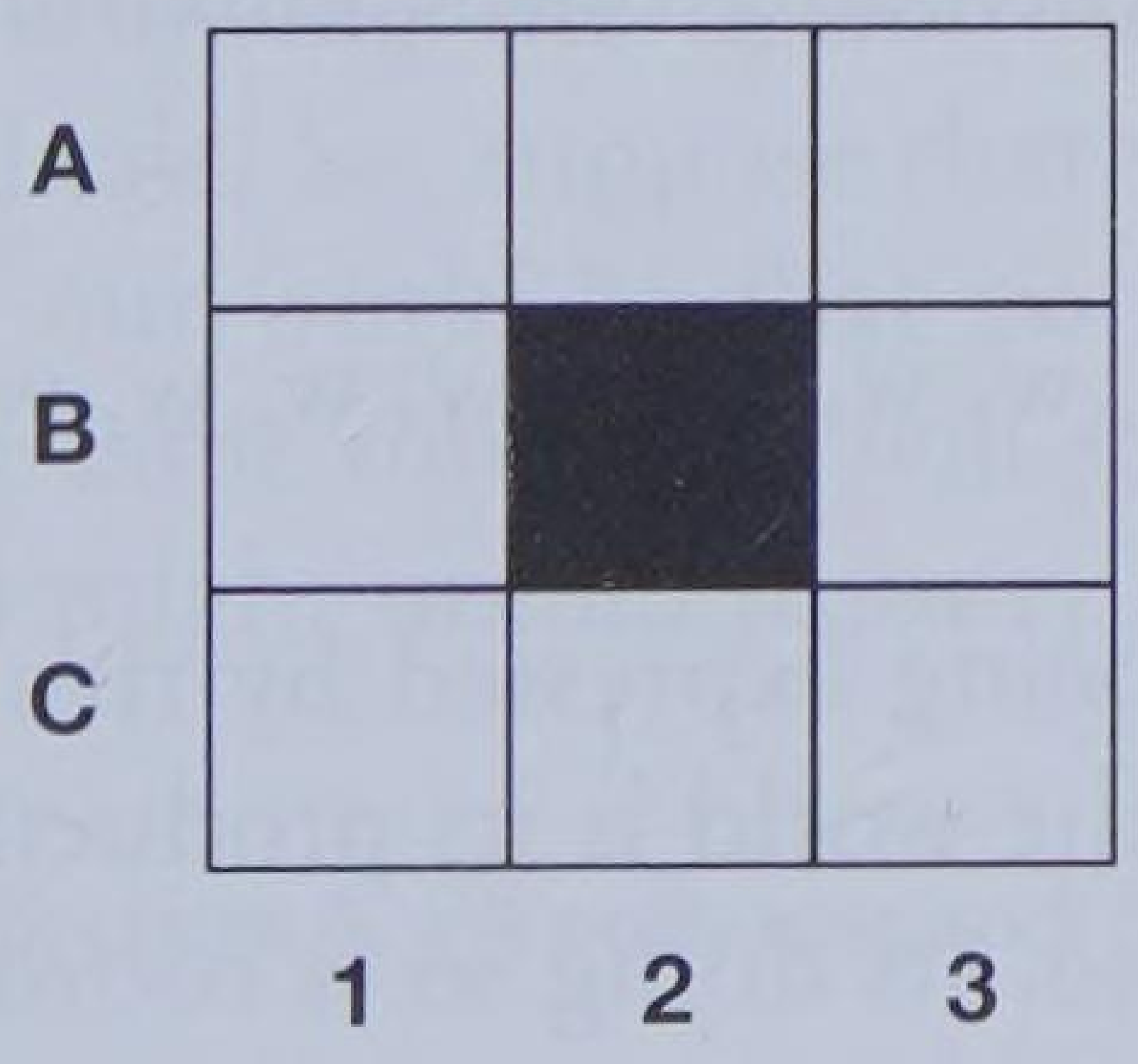

Figure 10.4

Supposing that we want the square to continue on its diagonal course upward, at t_3 our entry for B2 (and every other pixel with the exception of A3) will be white, and for A3 our entry will be black.

Understood in terms of their 'histories', of which each time-entry is a constituent, we could compare the pixels:

Table 10.1

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9
A1	w	w	w	w	b	w	w	w	w
A2	w	w	w	b	w	w	w	w	w
A3	w	w	b	w	w	w	w	w	w
B1	w	w	w	w	w	b	w	w	w
B2	w	b	w	w	w	w	w	w	w
B3	w	w	w	w	w	w	w	w	w
C1	b	w	w	w	w	w	b	w	w
C2	w	w	w	w	w	w	w	b	w
C3	w	w	w	w	w	w	w	w	b

So, A1, A2, and so on, are individuated according to their unique time-entry histories (we might add their positions relative to one another). No two pixels are alike. To be sure, each by itself is simply a history of a duration of white with an occasional 'blip' of black, but as a *system*, which is constituted in part by the information that relates the pixels to one another (here that information in part is to be found in their relative positions), produces our moving black square.

We might, for instance, produce the notion of A1, which determines a unique ordering of w's and b's (white and black entries). Thus, A1's notion would determine:

$$A1 = \{w, w, w, w, b, w, w, w, w, \ldots, w\}$$

I want to emphasize A1 as being expressed by this unique history, determined by its notion. Its activity in the world is its producing or organizing this unique history of w's and b's. Even so, its doing so is coordinated or 'harmonized' with the activities (and histories) of the other pixels. Were this not the case, there would be no unified *moving* black square, but a world of what would appear to be random blips of disconnected black squares.

In the *Discourse*, Leibniz says:

> the quality of being a king, which belongs to Alexander the Great, is an abstraction from the subject, and so is not sufficiently determinate to the individual, and does not involve the other qualities of the same subject, nor everything which the notion of that prince includes.[20]

I take this to suggest that there is a notion for being a king, but as such notions go, it is not sufficient for determining an individual. The same, I take it, would hold for the notions of a son, student of Aristotle, human being, and the like, all of which are notions constituting the notion of Alexander, but would also constitute the notions determining other individuals (so, the unique set and *ordering* of these functions, one to another, will also be part of the story of individuation). The mathematical analogue of the notion of Alexander being constituted in part by these other notions is a function constituted of other functions.

In terms of our primitive computer screen analogy, the notion of the black square is in part constituted of the notions of C1, B2, and so on. The notion of the black square would tell us how these other notions are related to one another, that is, how they relate in forming the notion of the moving black square. As we also know from Section 8 of the *Discourse*, Leibniz says that not only does an understanding of Alexander's notion account for how God knows Alexander's complete history, but suggests that it allows God to know the histories of every other individual substance. That is, from any moment of Alexander's history, God can know the state of the entire cosmos at any given moment. Likewise, if understanding the notion of the black square allowed us to know the *complete* history of the black square, and included in this notion are the notions of C1, B2, and so on, we will also know the complete histories of those very pixels. And, if the notions of those pixels are also included in the notions of other individual substances (other than the black square), we will in turn have epistemic access to the complete histories of these other individual substances.

In introducing another object into our world, then, we can make further sense of Leibniz's claim that, in knowing the notion of Alexander, God knows the histories of the other individuals. So, suppose that we have a second square, this one grey. And, let a *rule* of our world be that no two (non-white) squares can occupy the same pixel at the same time (this will serve to express the concept of impenetrability). Our world might unfold from t_1 to t_2 as follows:

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[20] *Discourse*, Section 8, (Woolhouse and Francks), p. 60.

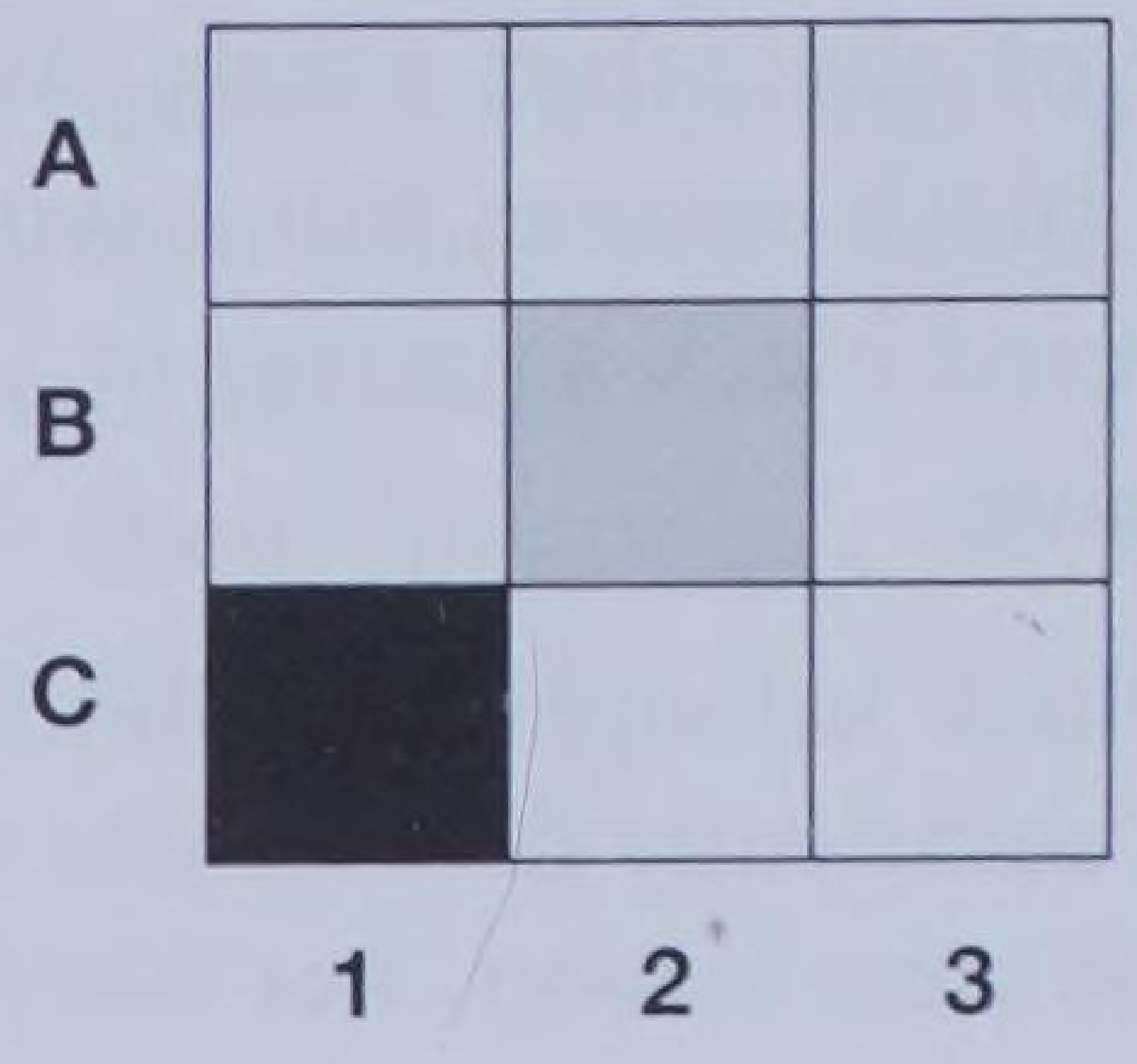

Figure 10.5

At t_1, the history of the black square is essentially connected to C1, whereas the history of the grey square is essentially connected to B2. Clearly, if two squares of the same colour maintained identical histories, nothing would turn on our saying 'two' here (Leibniz's principle of indiscernibles). Now, let the world unfold at t_2 as follows:

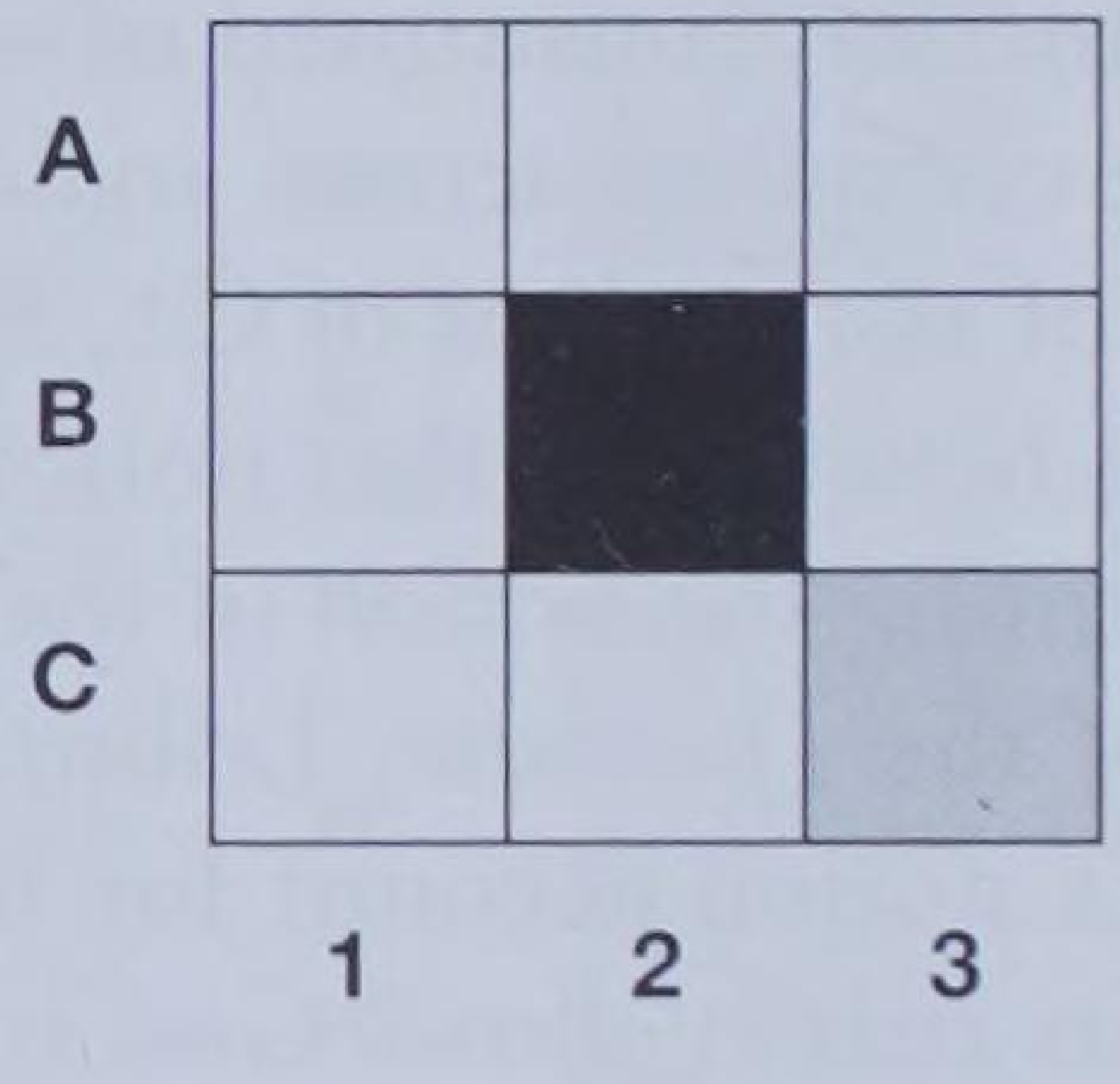

Figure 10.6

The notion of B2 is a constituent of the notions of both the black (at t_2) and grey (at t_1) squares. And so, if understanding the notion of the grey square allows us epistemic access to the notion of B2, and the notion of B2 is also a constituent of the notion of the black square, then in understanding the notion of the grey square we have epistemic access to the notion of the black square. So, if the black and grey squares are analogues to individual substances, we can see how it is that Leibniz could think that in God's knowing the notion of Alexander, God has epistemic access to the notions of other individuals.

We could certainly extend our analysis to include, in addition to Alexander, not only other human beings, but other objects such as tables, chairs, chariots, and even specks of dust, blades of grass, and so on. So, supposing that Alexander sits at a particular table, sitting on a particular chair, then it is in Alexander's notion that he do so. Moreover, Alexander's sitting is equally in the notions of this table and chair. For, it is in the history of this chair, for instance, that Alexander sat on it at this particular moment. And, as Aristotle looks at Alexander sitting, it is in Aristotle's notion that he sees Alexander sitting at this table and on this chair. Consequently, it is in the notions of Alexander, this table and this chair that they be seen by Aristotle at this moment in their histories. Thus, there is an interconnectedness holding between all the notions involved. And, since the notions of these other individuals will share notions constituting the notions of yet other individuals, God can in principle know (through a kind of metaphysical atemporal leapfrog, jumping from one notion to the next) the state of *all* individuals at any given moment in the cosmos. This appears to be connected to his later view in which he holds that every (state of every) monad mirrors the state of every monad in its universe. More on this connection below.

In the *Monadology*, Leibniz says, 'Every portion of matter can be thought of as a garden full of plants.'[21] It is not clear to me what he means by 'matter' here. I suspect that it is not simply Cartesian extension, but what else it may be is not clear. He says something similar in response to remarks made by Foucher (1695), which I think is of some exegetical help. There, in drawing a distinction between the *unity* of a material substance and the fact that its material nature requires it to be *infinitely divisible*, Leibniz says:

> In fact, the number of simple substances which make up a mass, however small that mass may be, is infinite, since in addition to the soul which constitutes the real unity of an animal, the body of a sheep (for example) is actually subdivided: that is to say, it too is an assemblage of invisible animals or plants, which themselves are also compounded, in addition to also having that which makes up their own real unity. And even though this goes on to infinity, it is obvious that, all in all, everything comes down to these unities, all the rest or the resultants being only well-founded phenomena.[22]

As others have suggested, the reference to animals and plants is very likely connected to his acquaintance with observations and studies of organic tissue (but also of pond water and other such things) under a microscope. But even here scholars, such as Daniel Garber and Catherine Wilson, have stressed that his point is that he sees even *these* entities to be 'built' out of smaller organisms, which in turn are 'built' out of smaller organisms, and so on ad infinitum. Garber showing that at one point Leibniz appeared willing to think of all material objects as being composed of a hierarchy of microscopic animals; Wilson arguing that the *idea* of the monad as 'living atom' appeared to stem from what Leibniz took the microscope to reveal.[23]

Leibniz's remarks, of course, are not to be confused with some sort of biological theory of *life*. The point, in line with what Garber and Wilson say, is not about biology or microscopic life per se, but about the *directedness* of activity (harmonious activity) that we observe in the 'building up' of complex material systems (organisms), informed regions of body working towards the preservation of structure over time. The unifying principle—that is, that which is responsible for the directedness—is not something to be found *in* corporeality or extension *simpliciter*, at least this is exactly what his earlier view, drawn out by Mercer, established. Rather, that principle is incorporeal. It is the monad (or interconnected system of monads) that is responsible for the directedness, which is brought out in his later view. And, of course, ultimately the monads acquire their directedness from God.

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[21] *Monadology*, proposition 67, (Woolhouse and Francks), p. 277.

[22] 'Remarks on M. Foucher's Objections (1695)', in Woolhouse and Francks, p. 185.

[23] Catherine Wilson, 'Leibniz and the Animalcula', in *Oxford Studies in the History of Philosophy*, M. A. Stewart (ed.) (Oxford: Clarendon Press, 1997), pp. 153–75.

Descartes himself even seems to have entertained the idea of thinking of the origin or principle of motion as being 'soul-like'. To More he writes:

> there are two different principles causing our movements. The first is purely mechanical and corporeal, and depends solely on the force of the spirits and the structure (*conformatione*) of our organs, and can be called *the corporeal soul*. The other, an incorporeal principle, is the mind or that soul which I have defined as a thinking substance.[24]

Three years earlier, to Mesland he writes that 'the numerical identity of the body of a man does not depend on its matter, but on its form, which is the soul'.[25] So, like Leibniz's remark in the above passage (that the soul accounts for the real unity of an animal), Descartes will not want to attribute the unity of a human body to corporeality, but to the mind. Even so, this does not preclude Descartes from at least entertaining the idea that organized body (such as a clock or an animal),[26] as opposed to extension *simpliciter*, is fully *informed* (i.e., *structured*). As he suggests in the above passage about the corporeal soul, the clockwork-like organization of internal bodily organs, and what we might call the *involuntary* movements of men's bodies that result from their organs being so configured, could arise from the very *structure* of the organs themselves. But, he appears to reject any connection between the way in which a structure might be said to account for the movements of a human body and the way in which a mind might be said to account for such movements.

In a letter to Mersenne, the year he writes the above letter to Mesland, again showing concern about the question over the origin of motion (*motuum principium*) in bodies, Descartes considers the possibility that such motion originates by way of a particle's having its own soul (that is, mind)—or even many souls. He expresses an additional worry, namely, the challenge of accounting for the harmonious motions of bodies. And yet, in the end Descartes is ready to reject the idea of assigning a soul (or many souls) to each particle of matter. In writing about the 'absurdity' of thinking that minds reside in particles of matter, which direct their harmonious activity, Descartes writes:

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[24] Letter to More, 5 February 1649 (AT V, 276; CSMK III, 365). Italics mine.

[25] Letter to Mesland, 1645 or 1646 (AT IV, 346; CSMK III, 279).

[26] In a letter to Morin, 12 September 1638, Descartes discusses the 'form' of a clock, which can account for its telling time (AT II, 367; CSMK III, 121–2). In a 23 November 1646 letter to the Marquess of Newcastle, Descartes likens the clock (its organized body) to animals (AT IV, 574–6; CSMK III, 303–4).

> For in order to make sense of this, one has to suppose not only that each particle of matter has a soul, and indeed several different souls, which do not impede each other, but also that these souls are capable of thought, and indeed divine, to enable them to know without an intermediary what was happening in those distant places[27]

As Leibniz would do later, Descartes worries over the difficulties in providing an account of the possibility of harmonious motion—an account of how the motions of bodies in the plenum work in harmony, together.

Just before making the above remarks about the bodies of animals, Leibniz introduces a metaphor of the unity we understand to hold among an infinite number of points that constitute a line:

> Those for whom lines are made up out of points have quite mistakenly looked for primary elements in ideal things, or in relations; and those who realized that relations like number, or space (which comprises the system or relations of possible coexistent things), could never be formed by the putting together of points, have for the most part then gone wrong by saying that substantial realities have no basic elements, as if they had no primary unities, and there are no simple substances. However, number and line are not *imaginary* things, even though they are indeed not made up in that way, because they are relations which involve eternal truths, in accordance with which the phenomena of nature are structured.[28]

For *ideal* things, he says, 'the whole is prior to the part', whereas for *real* things 'the whole is only a result, or assemblage' of the parts.[29] Leibniz stresses that number and line are not imaginary. Rather, line and point are *ideal*, which for Leibniz seems to mean in part that they determine the *possibilities* that underlie what is real.[30] The *real*, on the other hand, refers to what is *actually* the case. We saw a distinction like this earlier, when Descartes drew the distinction between a mathematical physics and pure mathematics. The former deals with how bodies *actually* stand to one another, the latter with the *possibility* of the arrangement of bodies.

To think that a line is 'built up' out of points, Leibniz suggests, is wrongheaded. A line is extended, points are not, and so one simply cannot 'build' a line out of points—the points cannot be set side by side so as to form a string, the string of points being the line. The line, he says, is the *whole*, and for ideal entities like lines and points, the whole is prior to the parts. So, the line is prior in an important sense to points—even though the points *are* said to be the elements of the line. A question that arises at this point is whether what Leibniz is driving at here is a logical or conceptual claim, rather than an ontological one (though my question should not be taken as ruling out the possibility that it is ontological). What interests me here is whether he is telling us that points are *unintelligible* conceived independently of extension. If so, then the intelligibility of a point requires the conception of extension, in which case the whole is (conceptually) prior to the part. However, recognizing this, it seems to me, he is not precluded from going on to say that points can be understood as constituting the line as its elements. Here we might want to draw a distinction between parts and elements. A *part* of a line is a line, but an *element* of a line is not. Leibniz has provided the very conceptual mechanism to help us understand how points make up a line—lines are made up of points, he says, but not in the *way* that a sheep's body is made up of smaller organisms. As he puts it, 'there is composition only in *concrete* things, or *masses*'.[31] So, in contrast to the sheep's body, which is a concrete thing or mass, a line is not a composition or aggregate of points. Rather, a line is a specific *relation* of points, he says, a relation involving eternal truths. One way to express or represent that relation, I take it, is by way of the linear function (or to use his earlier terminology, *notion*). The function *shows* us exactly how the points are related, and is thus the notion expressing that line.

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[27] Letter to Mersenne, 20 April 1646 (AT IV, 401; CSMK III, 285).

[28] *Leibinz's Remarks to Foucher* (Woolhouse and Francks), p. 185.

[29] Ibid.

[30] Ibid., pp. 184–5.

Seen through the lens of the earlier view, *matter*, in the remark about its being a garden full of plants, would be taken to mean *corporeal substance*. Such a substance, recall, arises as a result of mind's acting through the metaphysical medium of matter (which for Leibniz is extension *simpliciter*). But, as commentators have argued, extension or matter in Leibniz's later view is supposed to be reduced ultimately to the monads, which are dimensionless. Matter is not something that exists independently of the activity of the monads. Rather, matter arises as the set of *relations* that monads have to one another. But even in this later view, it seems to me, the possibility of a body or material substance depends on that *relational nexus* of monads. And, if matter (extension) is this nexus, and body or material substance depends on matter, then matter is prior to body or material substance.[32] Keeping things as clear as we can, then, I will continue to call Leibnizian corporeal or material substance 'body', and that which metaphysically underlies body 'matter'. Matter is taken to be metaphysically prior to body from here on out. Leibnizian matter is Cartesian extension, whereas Leibnizian body is not. Leibnizian body is *informed extension*, which, as we saw with the pixel example, in part means that it is *impenetrable extension*. This, it seems to me, is compatible with both Leibniz's earlier and later views.

Thus, even though the sheep's body is an aggregate of smaller organisms, which in turn are aggregates of smaller organisms (ad infinitum), if the sheep is an individual substance, its being such is determined by that which directs the bodies involved in the aggregate towards the preservation of this and no other structure. And, that principle is nowhere to be found independent of the organized regions of *body*. Even so, this principle of activity is not found in or derivative of *matter*. And so, even though Leibniz may have those smaller organisms in mind (those observed through microscopes) when referring to matter's being 'a garden full of plants', I contend that in insisting on the early view, he should be taken to be referring to corporeal substance or body, or in insisting on the later view, he should be taken to be referring to bodies arising in part from that relational nexus—but either way, we can think of those 'bodies' as matter informed by mind. Read this way, since monads are the origin of activity, diversity, and harmony in the cosmos, the *monads* are best taken as the metaphysical analogues to those plants—fertile and directed in their activity—which accounts for the organization of the material cosmos at any perceivable level. Along the lines of the sheep case, then, Leibniz can be taken to be emphasizing that with respect to the cosmos, any perceived region of it (the phenomena), regardless of how small, is *completely* informed. There simply is no body or body part (regardless of size) in the cosmos that exists as an inert expanse of Cartesian extension, independent of the activity of mind. In other words, unlike body, matter will never be observed in the cosmos. This does not entail that matter does not exist. For, the account here is taking matter as metaphysically prior to body. So, if there is body (which *is* observable), there is matter (which is not observable). Matter is a necessary (though not a sufficient) condition for body.

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[31] *Leibinz's Remarks to Foucher* (Woolhouse and Francks), p. 184.

[32] This, I believe, is along the lines of Vincenzo De Risi's view. De Risi, 'Leibniz on Geometry: Two Unpublished Texts with Translation and Commentary', in *The Leibniz Review*, vol. 15 (December 2005), pp. 127–51. He suggests the view on pp. 129–30.

In Proposition 52 of the *Monadology*, Leibniz says:

> And this is the way in which there is mutual action and passivity between created things. God, in comparing two simple substances, finds in each of them reasons which oblige him to adapt the other one to it, and as a result the one which is active in certain respects is passive with regard to a different point of comparison.[33]

Looking back to our world of moving squares, he seems to be saying that God, when comparing the black and grey squares, has reasons that oblige him to move the grey square away from B2 as the black square comes to occupy B2 (the obligation, for example, could come from the rule that no two non-white squares can occupy the same pixel space at the same time, where the 'pixels' are (as we will see below) best taken to be *logical spaces* that are the locus of information in the system—consequently, *being white*, *black*, and *grey* are information that 'fill' these spaces).

In proposition 56, he says:

> Now, this *interconnection*, or this adapting of all created things to each one, and of each one to all the others, means that each simple substance has relationships which express all the others, and that it is therefore a perpetual living mirror of the universe.[34]

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[33] Leibniz (Woolhouse and Francks), p. 275.

[34] Ibid.

And, referring back to Bayle's criticism, in proposition 59, he says:

> But he [Bayle] was unable to bring forward any reason why this universal harmony, which means that every substance exactly expresses every other through the relationships it has with them, was impossible.[35]

Given the above sorts of passages, I think that the computer screen analogy can be extended to cover the later view that monads are the substances of the cosmos, while not doing damage to its being used to cover the earlier view that offered Alexander the Great as an example of an individual substance. The obvious analogue to the monad is the pixel. This is a bit dangerous (interpretively speaking), of course, since pixels *are* spatially locatable. But, it is not their spatial location on the screen that I want to emphasize (though we will have to consider this aspect as information). What make the pixels important, I think, are the activity, diversity, and harmony they exhibit when the computer screen is turned on and working. The activity, diversity, and harmony are indicated by way of *information—programming* to be more precise. If only one pixel lit up and randomly changed the occasional colour, the computer screen would be useless. What makes it something worth looking at is when all the pixels 'work together' and are lit up as a system. The system of pixels is a system of information, organized by way of a program. It is this *informational* feature that I want to emphasize in making monads analogues to pixels.

What I have in mind is not that each pixel understood simply as a bubble occupying some physical location *in* the glass screen is a monad, though it could be. In my view, it will not matter whether one physical bubble is taken to be a monad or a ten-by-ten region of bubbles is a monad, or a thousand-by-thousand region of bubbles is a monad. So, as I indicated a moment ago, the monad will be the logical *locus* of the information, and more importantly, the *smallest unit* of information in the system. So, if that unit of information 'controls' or 'informs' one physical bubble, fine. If it controls or informs that thousand-by-thousand region of bubbles, then that is fine, too. The point is that if the latter is the case, there is absolutely no distinction (in terms of information) between any of the physical bubbles constituting the thousand-by-thousand area. Any region smaller than a thousand-by-thousand area of bubbles will be informationally speaking meaningless. So, in my view, if a monad is the analogue to the pixel, and a monad controls or determines a given thousand-by-thousand region of physical bubbles, there is nothing stopping us from calling the thousand-by-thousand *region* of bubbles *a* pixel (so, the idea of the single physical glass bubble is irrelevant to what counts as a pixel—the pixel will be whatever is the region of screen controlled by the smallest unit of program information).

In our primitive computer screen case, A1, B3, and so on, are pixels. Letting them serve as analogues to monads, we can see how the monads work to produce the world of the *moving* black and grey squares. From one point of view, namely the point of view of the programmer, one can produce our world of moving squares by simply programming the pixels. So, at t_1, pixel C1 is programmed to exhibit black, pixel B2 is programmed to exhibit grey, and all others are programmed to exhibit white. At t_2, pixel B2 is programmed to exhibit black, pixel C3 is programmed to exhibit grey, and all others are programmed to exhibit white. Let's call this the *bottom-up* point of view, where by this I mean that we account for the world of moving squares from the bottom (the pixels) up (to the moving black and grey squares). This would be analogous, I think, to accounting for the world from the bottom (the monads) up (to individuals such as Alexander). Even so, in suggesting this I am not saying that Leibniz's earlier and later views *must* be brought together. Rather, I am only suggesting that the analogy will work for both views.

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[35] Leibniz (Woolhouse and Francks), p. 276.

Of course, in order for our programmer to know which pixels are programmed black, which grey, and which white, at any given time, requires a knowledge of certain rules—such as no two non-white squares can occupy the same pixel space at the same time. In light of this, another way to approach the programming of our world is to formulate the notions (or functions) describing the movements of our black and grey squares. Like linear functions, which are able to take any x value on the plane and pair it with a y value such that the ordered pair will map onto the lines determined by those functions, the notion of the black square, for instance, would take any A, B, or C, and pair it with a 1, 2, or 3, such that the pair would map the location of the black square at that time in its history. We can think of our world as being ordered from the *top-down*—where by this I mean that we account for the world of moving squares from the top (the *moving* squares) down (to the pixels). This, then, is analogous to accounting for the world from the top (the notion of Alexander) down (to the monads). Both the *bottom-up* and *top-down* points of view (programming-wise) produce the same world of moving squares.

Now, in thinking about the world in which two squares are moving, where each is expressed as a linear function, both equations (functions) will have to be solved simultaneously. In other words, a *system* of equations will have to be solved. Does our analogy commit Leibniz conceptually to more than he can handle? That is, is the solving of a system of equations something with which Leibniz can conceptually deal? The answer to this last question is, yes. In fact, thinking of notions as equations (that define functions) sheds some light on what I think was Leibniz's interest in what are called *determinants*.

George Simmons notes that 'the role of determinants in elementary algebra is a good illustration of Marshall's Generalized Iceberg Theorem: seven-eights of everything cannot be seen'.[36] The idea here is that if one is dealing with determinants as a mathematical apparatus to solve equations, there is much more going on mathematically than this apparatus exhibits on the face of it. And, not only does this theorem apply to determinants (and to most things mathematical), but it no doubt applies to Leibniz's *Monadology*. This relatively short work, it is safe to say, is just the tip of the philosophical iceberg. In the words of the Iceberg Theorem, seven-eighths of the work's mass is below the surface (and it is some of this that I am trying to expose by appealing to Leibniz's knowledge of certain mathematical concepts).

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[36] George F. Simmons, *Precalculus Mathematics in a Nutshell* (New York: Barnes & Noble, Inc., 1987), p. 71.

The view that there is more than meets the eye here is prompted in part by Leibniz himself—for about his own use of lines, notions, and equations in his *philosophical* discussions, he admits:

> I make use of these comparisons in order to sketch some imperfect picture of the divine wisdom, and to say something which might at least raise our minds to some sort of conception of what cannot be adequately expressed.[37]

In correspondence with the Marquis de L'Hospital (1694), Leibniz writes: 'My metaphysics is all mathematics, so to speak, or it can become so.'[38] And so, my relying here on mathematical concepts in making Leibniz's philosophical (and certainly metaphysical) view clearer is not off track. And so, I will continue in this vein.

In earlier letters to L'Hospital (though one letter is dated 1693, Eugene Smith believes some of the correspondence connected to determinants goes back to 1678),[39] Leibniz demonstrates how a system of linear equations can be solved simultaneously. He writes:

> [C]onsider three simple equations in two unknowns, the object being to eliminate the two unknowns and indeed by a general law. I suppose that
>
> $$10 + 11x + 12y = 0, \tag{1}$$
>
> and
>
> $$20 + 21x + 22y = 0, \tag{2}$$
>
> and
>
> $$30 + 31x + 32y = 0, \tag{3}$$
>
> where, in the pseudo number of two digits, the first tells me the equation in which it is found, the second, the letter to which it belongs. Thus on carrying out the computation,

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[37] *Discourse*, Section 6.

[38] Leibniz, Letter of 27 December 1694, in *Leibnizens mathematische Schriften*, C. I. Gerhardt (ed.), 7 vols, (Berline, 1846–1860). This is found in vol. 2, pp. 255–62. Vincenzo De Risi quotes a portion of this letter in his discussion of the geometrical work of Leibniz. De Risi (2005), p. 128.

[39] 'Leibniz: On Determinants', translated by Thomas Freeman Cope, in *Source Book in Mathematics*, edited by David Eugene Smith (New York: McGraw-Hill, 1929), pp. 267–70. Smith makes the remark in his preface on p. 267.

we find throughout a harmony which not only serves as a check but even makes us suspect at first glance some rules or theorems. For example, eliminating y first from the first and second equations, we shall have:

$$\begin{array}{c} 10.22 + 11.22x \\ -12.20 - 12.21x \ldots^{40} \end{array} = 0; \qquad (4)$$

Here, Leibniz is demonstrating to L'Hospital that he can use numbers instead of letters to keep track of the procedure. So, in looking at the coefficients of the first equation, namely 10, 11, 12, the first digit—that is, **1** in **1**0, **1**1, and **1**2—indicates that the coefficients belong to the first equation, the second digit—**0**, **1**, **2**, in 1**0**, 1**1**, 1**2** respectively—indicates the coefficients' locations with respect to the variables in the equation (0 indicates no variable, 1 indicates x, 2 indicates y). Thus, 11 indicates that the coefficient is from equation 1 and is the coefficient of x, 21 indicates that the coefficient is from equation 2 and is the coefficient of x, 22 indicates that the coefficient is from equation 2 and is the coefficient of y, 31 indicates that the coefficient is from equation 3 and is the coefficient of x, and so on.

In the above passage, I cite that part of the letter in which Leibniz eliminates y with respect to the first and second equations. In the letter, he also goes on to eliminate y with respect to the third equation. But, for what I am about to suggest, I need only show how the elimination procedure works, and showing how it works for the first and second equations is sufficient. In order to best explain what Leibniz's procedure is, let me first cast the determinant and the procedure of solving a system of equations in more familiar notation (which, we will see, is easily connectable to Leibniz's). Let's begin with the anatomy of a determinant of a square matrix.

A square matrix is an n x n array of elements a_{ij}, where i indicates a's row and j a's column. So, a_{23} tells us that in the matrix of which it is an element it is located at row 2, column 3. Consider a 3 x 3 matrix.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Figure 10.7

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[40] Leibniz, in D. Smith (1929), p. 268. In the translation, Cope does not include the variable x after 12.21 in equation (4). He simply provides an ellipsis. I have included it here for the sake of clarity.

Now, consider the simpler 2 x 2 matrix:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Figure 10.8

Let this be matrix A. The determinant of A is a number denoted by |A|. We can define the determinate of A as follows:

$$|A| = a_{11}\,a_{22} \;-\; a_{12}\,a_{21}.$$

The determinant of A is calculated by cross-multiplying a_{11} (which is the upper-left corner) with a_{22} (which is the lower-right corner), and subtracting from this the product of a_{12} (which is the upper-right corner) and a_{21} (which is the lower-left corner). Here is an example:

$$\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}$$

Figure 10.9

We find the determinant of this matrix by cross-multiplication, where we subtract the second product ($a_{12}\,a_{21}$) from the first ($a_{11}\,a_{22}$):

$$\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}$$

$$(1 \times 3) - (2 \times 4) = 3 - 8 = -5$$

Figure 10.10

So, the determinant of this matrix is −5.

What Leibniz sees is that determinants can be used to solve a system of equations. He says:

> Given any number of equations which is sufficient for eliminating the unknown quantities which do not exceed the first degree:—for the final equation are to be taken, first, all

possible combinations of coefficients, in which one coefficient only from each equation is to enter; secondly, those combinations, after they are placed on the same side of the final equation, have different signs if they have as many factors alike as is indicated by the number which is less by one than the number of unknown quantities: the rest have the same sign.[41]

In this passage, he notes that determinants can be used to solve equations that do not exceed the first degree, which means that the variables are not squared, cubed, and so on. Thus, here he is specifically talking about linear equations (though polynomials of higher degree could also be solved this way).

The procedure that Leibniz describes in the above letter can be explained as follows. Suppose that we have two equations with two unknowns:

$$a_{11}x + a_{12}y = b_1,$$
$$a_{21}x + a_{22}y = b_2.$$

The coefficient matrix is the 2 x 2 matrix above (see Figure 10.8). To eliminate y, we must first multiply the first equation by a_{22}, which is the y coefficient of the second equation:

$$a_{11}\ a_{22}x + a_{12}\ a_{22}y = b_1\ a_{22}.$$

Next, we must multiply the second equation by $-a_{12}$, which is the (negative of the) y coefficient of the first equation:

$$-a_{12}\ a_{21}x - a_{12}\ a_{22}y = -b_2\ a_{12}.$$

Next, we add the two equations. Notice that—$a_{12}\ a_{22}y$ of the second equation and $a_{12}\ a_{22}y$ of the first cancel out each other. What remains is:

$$a_{11}\ a_{22}x - a_{12}\ a_{21}x = b_1\ a_{22} - b_2\ a_{12}.$$

Now, we solve for x:

$$x = \frac{b_1\,a_{22} - b_2\,a_{12}}{a_{11}\ a_{22} - a_{12}\ a_{21},}$$

Leibniz's first equation, recall, is $10 + 11x + 12y = 0$, and his second equation is $20 + 21x + 22y = 0$. Here, he has set both equations equal to 0. Before setting them to equalling 0, the equations would have been in the more familiar form:$11x + 12y = -10$ and $21x + 22y = -20$. So, the counterparts to 10 and 20 in the explanation (in which we used more familiar terms) are b_1 and b_2 respectively. Now, to eliminate y, he multiplies the first equation by 22 (which is the y coefficient of the second equation) and multiplies the second equation by -12 (which is the negative of the y coefficient of the first equation). He writes this as follows:

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[41] Leibniz (D. Smith 1929), p. 269.

$$\begin{matrix} 10.22 + 11.22x \\ -12.20 - 12.21x \end{matrix} = 0$$

'10.22' is 10 x 22, '11.22' is 11 x 22, and so on. 22 is a_{22} in the above explanation of the procedure, and -12 is $-a_{12}$. Notice the similarity between Leibniz's '22' and the more familiar 'a_{22}', and his '-12' and the more familiar '$-a_{12}$'.

Eugene Smith makes the following assessment of what Leibniz says in the above letters:

> The letters to L'Hospital disclose the remarkable fact that, more than fifty years before the time of Cramer, who was the real moving spirit in the development of the theory, the fundamental idea of determinants had been clear to Leibniz and had been expounded by him in considerable detail.[42]

Gabriel Cramer (1704–1752) was no stranger to Leibniz's work. In fact, he was responsible for publishing the correspondence between Johann Bernoulli and Leibniz. He also published the complete works of Johann Bernoulli, and five volumes of the work of Christian Wolff.[43] Cramer, who together with Giovanni Ludovico Caladrini held the chair in mathematics at the Académie de Clavin in Geneva, travelled as part of his duties, and spent time with Johann and Daniel Bernoulli, and Leonhard Euler, to name just a few. It is in his *Introduction à lánalyse des lignes courbes algébraique* (1750) where he develops his now famous rule for solving a system of equations—called *Cramer's Rule*—which goes as follows.[44]

Consider again this system of equations:

$$a_{11}x + a_{12}y = b_1,$$
$$a_{21}x + a_{22}y = b_2.$$

We know the coefficient matrix is (see Figure 10.8): To find x, we substitute b_1 and b_2 for the x coefficients a_{11} and a_{21} respectively, and divide the new matrix by the above coefficient matrix. Thus:

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[42] Leibniz (D. Smith 1929), p. 267.

[43] I am relying on J. J. O'Connor and E. F. Robinson's brief online biography of Gabriel Cramer, at the time of writing available at http://www-history.mcs.st-andrews.ac.uk/Biographies/Cramer.html.

[44] O'Connor and Robinson, http://www-history.mcs.st-andrews.ac.uk/Biographies/Cramer.html.

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

Figure 10.11

The quotient to the far right is exactly what Leibniz arrives at in the letter to L'Hospital. To find *y*, we substitute b_1 and b_2 this time for the *y* coefficients a_{12} and a_{22} respectively,

$$y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

Figure 10.12

Carl Boyer notes that the rule for solving simultaneous equations by determinants appeared as early as in Colin Maclaurin's 1748 *Treatise of Algebra*, published two years after Maclaurin's death, but two years before Cramer's book.[45] Boyer also notes something worth keeping in mind, namely that 'the history of determinants is an oddly chequered one, with sporadic bits of the subject appearing in ancient China, some work by Leibniz, a rule of Cramer, and some symmetric results of Lagrange'.[46] Be that as it may, as Boyer and others point out, what little Leibniz says about determinants shows that he commanded a rather sophisticated grasp of the concept.

Any system of equations, even a large system (where certain conditions are met), in principle can be solved. What this means is that in one fell swoop—that is *simultaneously* by way of determinants—we could find values for *x* and *y* that satisfy every equation of the system (that is, of course, the equations describe lines that intersect on the plane). Extending this idea to notions, which are understood by analogy as functions (defined by equations), and which are understood as determining individual substances in the cosmos, in principle, values for the 'variables' common to all notions can be found, values that would satisfy the variables of each and every notion (this assumes that the notions define substances that encounter one another in the world). In this way, we get *some* sense of how God is able to 'harmonize' the individuals of the cosmos. Something like this seems to be running through Leibniz's mind. And, the use of mathematics here to shed light on such an insight is compatible with his earlier admission that he uses mathematics 'to sketch some imperfect picture of the divine wisdom, and to say something which might at least raise our minds to some sort of conception of what cannot be adequately expressed'.[47]

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[45] Boyer (1985), p. 471. [46] Ibid., p. 557.

If what I have said is right, then we might draw from his work on determinants and functions the following lesson by analogy: the divine wisdom, in its thinking out the cosmos, is something like a matrix program, which 'harmonizes' the activities of all individuals (the monads or things, such as Alexander the Great—here is where it does not matter which), in much the same way that a determinant (simultaneously) solves a system of equations.

To shed further light on what I take Leibniz to have been thinking, several examples come to mind. For instance, matrix algebra is used to solve a system of equations that define a *single* object, where the object is taken as 'moving' across the plane. Such systems are sometimes referred to as *transformations*. This is very difficult mathematical subject-matter, and fortunately for me (and for the reader) it is something that need only be briefly discussed in order for me to make my point. Let's begin by considering the illustration below.

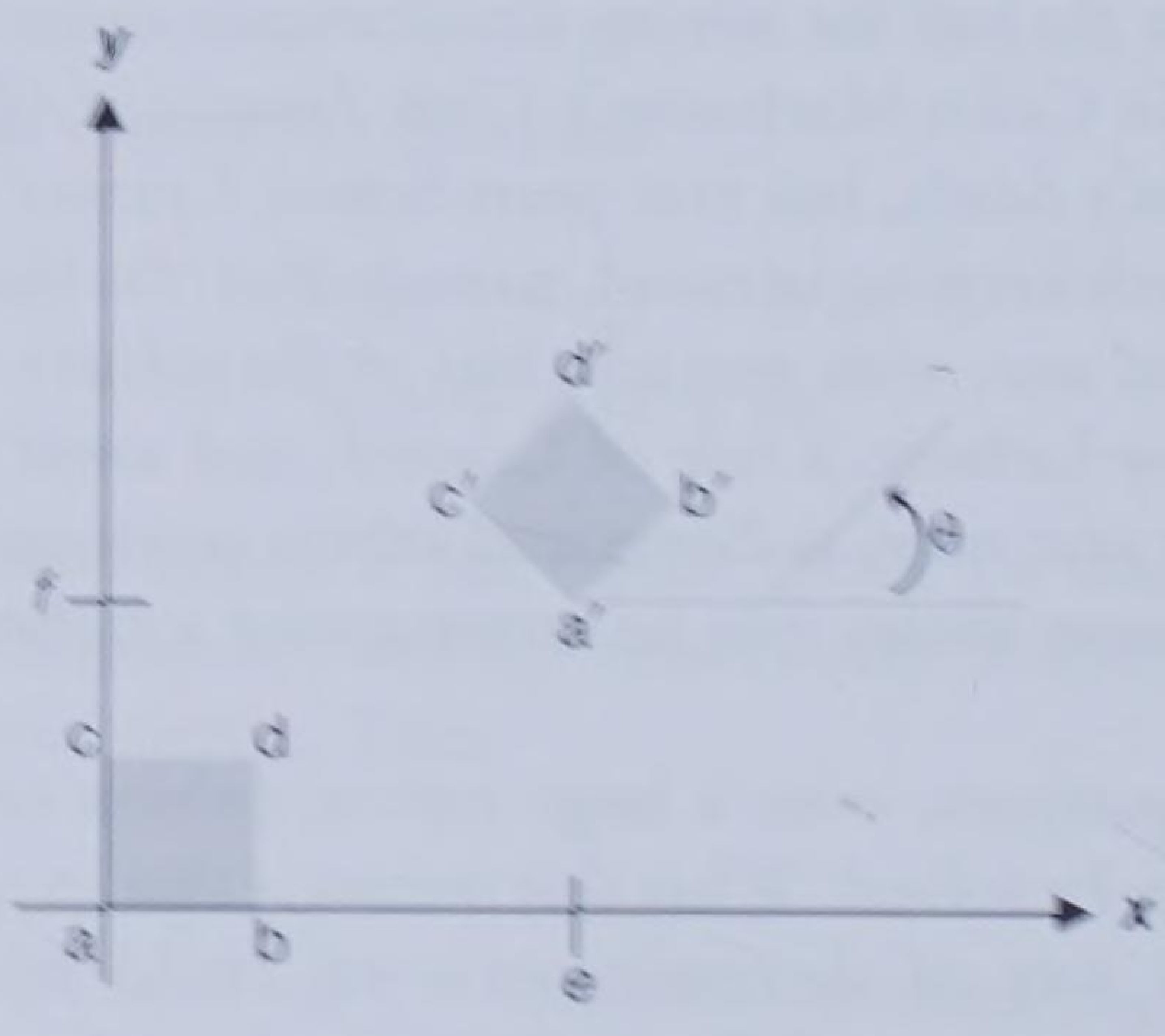

Figure 10.13

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[47] Leibniz, *Discourse*, Section 6, Woolhouse (1998), p. 58.

Letting the square in the lower-left corner be our *object* (it is what is sometimes called the *source* object, so in this case, the source square), where vertex *a* is located at (0, 0), what we will need to do (if we want to move it) is keep the square *qua object* intact (mathematically speaking).

This will require us to keep the four corners (vertices) *a*, *b*, *c*, *d*, for example, at the right distances from one another (or, the lengths of the sides *ab*, *ac*, *bd*, *cd* must be kept equal in length), the interior angles of sides *ab* and *ac*, *ac* and *cd*, *ab* and *bd*, *bd* and *cd* must be maintained (which are right angles), and so on. The square higher and to the right of our source square is the *new* location of the square, after moving it. There is a function $F(x)$ such that $F(a) = d'$, and so on for each vertex, though here it is possible to focus on functions that will relocate entire sides; f and e represent the distances our square is moved up and to the right; and θ represents the angle our square is rotated (counter-clockwise). To be sure, a lot to juggle here, but matrix algebra allows us to do just this. Though more is contained in it than meets the eye (thinking again of that Iceberg Theorem), the equation looks rather simple. Where T is a transformation function:

$$T\left(\begin{vmatrix} x \\ y \end{vmatrix}\right) = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} + \begin{vmatrix} e \\ f \end{vmatrix}$$

Figure 10.14

This matrix formula will keep our square intact, where we can relocate it anywhere on the plane we wish. Unpacking this a bit can help us see the functions involved:

$$T\left(\begin{vmatrix} x \\ y \end{vmatrix}\right) = \begin{vmatrix} x\cos\theta - y\sin\theta + e \\ x\sin\theta + y\cos\theta + f \end{vmatrix}$$

Figure 10.15

In assigning 0 to both x and y, the transformation function T tells us that the new coordinates are e and f respectively. Looking at the graph above, this means that the vertex *a* of our source square, located at (0, 0) is moved to its new coordinate place (e, f), where e is the shift to the right on the x axis and f a shift up on the y axis. The idea is that for any x and y pair in the source square, the transformation function T will map us onto exactly one x and y pair in the square's new location.

The first of the above formulations of T is set up so that we could solve it by way of Leibniz's or Cramer's procedure for dealing with determinants. This sort of thing was not beyond Leibniz's understanding of early matrix algebra. In fact, one of the earliest remarks about transformations of this kind is made by

Leibniz, though instead of a geometrical figure, such as the square considered in our example, he considers the transformation of a geometrical line.[48] De Risi finds in both Leibniz's *Ars Representatoria* (1691) and *Uniformus Locus* (1692) that Leibniz entertains the idea of transformations of geometrical figures.[49] So, the *idea* of a transformation was something he grasped.

Here is another example that can help to illustrate Leibniz's insight. A weather report on television included an item about global wind currents. The computer graphic used in the report showed the current on the Earth's surface, the current represented by vectors. The graphic represented the dynamic changes in the wind's current by exhibiting a systematic change in the lengths and directions of the vectors. Each vector on the surface originated from a point on the surface, where that point did not move. What changed was the arrow's direction and length. As would be expected of such a representation, the arrow's direction represented the wind's direction and the arrow's length represented the wind's speed (together they represented the wind's velocity). And so, if the change in wind current were such that it decreased in speed, the arrow's length decreased in proportion. And, if the change in wind current were such that it increased in speed, the arrow's length increased. If the current changed direction, the arrow pointed in the new direction. Now, given that the global surface is bounded, as vectors in one region of the surface increased in length, other vectors had to decrease in proportion. And, as vectors in one region changed direction, other vectors had to change their direction—the changes occurring simultaneously.

The graphic, I think, is a nice example of the sort of thing that Leibniz is getting at in the *Monadology*. The monads are the system of informed points constituting the sphere's surface. An equation (which is information in this case) determines a vector that originates from a point. The system of equations is in fact solved by a matrix formula, which works to produce the 'harmony' (the simultaneous changes) observed among the different vectors. This is from a programmer's perspective.

The weather graphic was supported by an underlying mathematical structure, which I earlier called a *vector space*. The graphic was made possible by way of some set of objects operated over by two binary operators. There are only two mathematical structures that can underwrite a vector space—a ring and a field. A Boolean algebra is simply not enough. As noted earlier, a ring is a system of a set of objects operated over by two operators of combination, identity elements for each operator, but inverses with respect to only one of the operators. It is a system constituted of a monoid and a group (the latter having the inverses). A field is a system of a set of objects operated over by two operators of combination, identity elements for each operator, and inverses with respect to both operators. It is a system constituted of two groups. All fields meet the criteria for being rings, but not the other way around. If Leibniz's description of the cosmos in the *Monadology* is similar in the right sorts of ways to the weather graphic (and I think that it is), the mathematical structure that would make such a cosmos *possible* (that is, the one described in the *Monadology*) is either a ring or a field. Like I said, a Boolean algebra is not enough.

---

[48] Benoit Mandelbrot notes this in his *Fractals and Chaos* (New York: Springer, 2004), p. 206. Although Mandelbrot does not say where Leibniz makes the remark, I believe that it is in a mathematical fragment, page 547, of vol. 1 of *Leibnizens mathematische Schriften*, C. I. Gerhardt (ed.), 7 vols (Berline, 1846–1860). In the fragment, Leibniz writes: '*Linea est extension quod describitur motu puncti*', which I translate as, 'A line is the extension in which the motion of a point is described', or better (reworking the passive *describitur*), 'A line is the extension describing the motion of a point.' Following this, he introduces the idea of '*transformationem seu motum*', which is talk about the transformation of the line that describes a point's motion.

[49] Vincenzo De Risi (2005), pp. 135, 147.

We have now arrived at exactly where we wanted (when first embarking on this discussion). To be sure, Leibniz may be unaware of the extent to which the underlying mathematical structure (which makes his metaphysical picture of the cosmos possible) differs from the structure we are calling a Boolean algebra, but his description of the cosmos in the *Monadology* is nevertheless conceptually underwritten by such a structure. What I am suggesting is that, given his sophisticated grasp of those things that go into understanding such structures—equations, functions, determinants, scalars, transformations, and so on—and given his admission of the use of such concepts in the clarification of his philosophical work, it is not far fetched to think that in the *Monadology* he has in mind a world that is, plain and simple, an ideal mathematical structure, a structure that underwrites the very *possibility* of the actual world—where that structure, as it turns out, is either a ring or a field. Either way, the *Monadology* describes a world whose mathematical structure is more sophisticated than a Boolean algebra.

Let's move on to a related topic before bringing this part to a close. The topic is the *use* of informed matter (i.e., body) to assist us in calculating—specifically in adding and multiplying. Combinatorics played an important role in developing what today we call 'computation'. Seventeenth-century calculating machines bear the marks of the philosophical connection made between the combinatorial art and the basic operations of arithmetic—most notably addition and multiplication. Taking a brief look at what went into the development of a few of these machines will help us better understand how the connection was made. What is more, it is within this context that the metaphysical view—that matter is importantly connected to instantiating (or manifesting) the conditions necessary and sufficient for the basic operations of arithmetic—comes to the surface.

11

Material Structure and Calculating Machines

One of the insights into finite groups, or mathematical structures, is that they are cycles or closed systems. Consider something called the *square group*.[1] Let's start with a square:

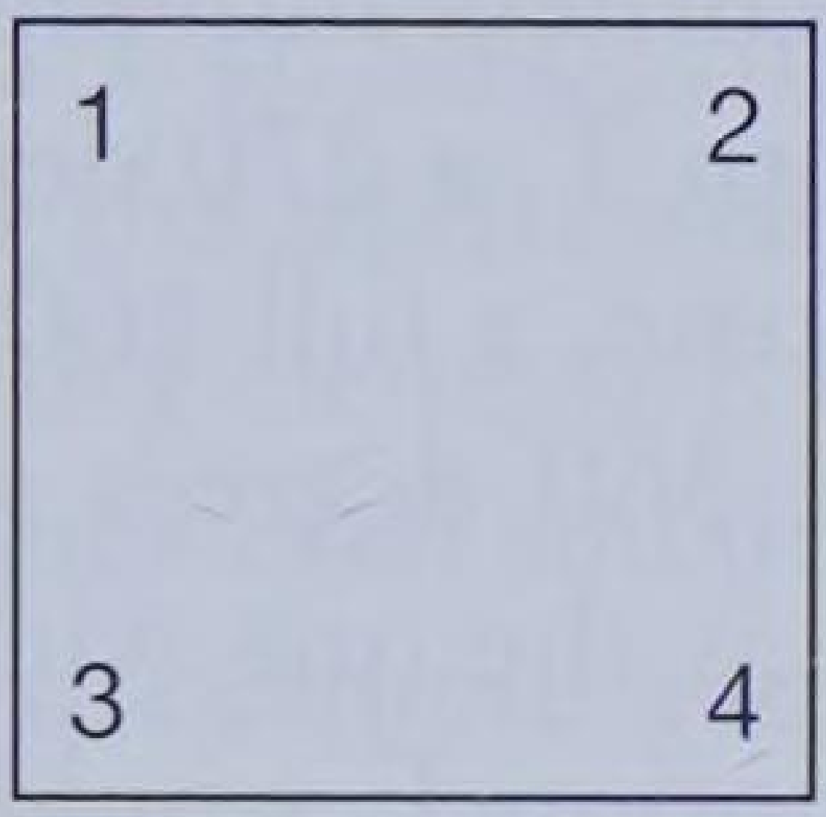

Figure 11.1

The corners have been labelled 1, 2, 3, and 4, to help keep track of 'moving' the square around. Now, let's 'move' the square by first rotating it around its centre. Rotation movements will be in a counter-clockwise direction. If we rotated the square a quarter turn (90 degrees), it would look like this:

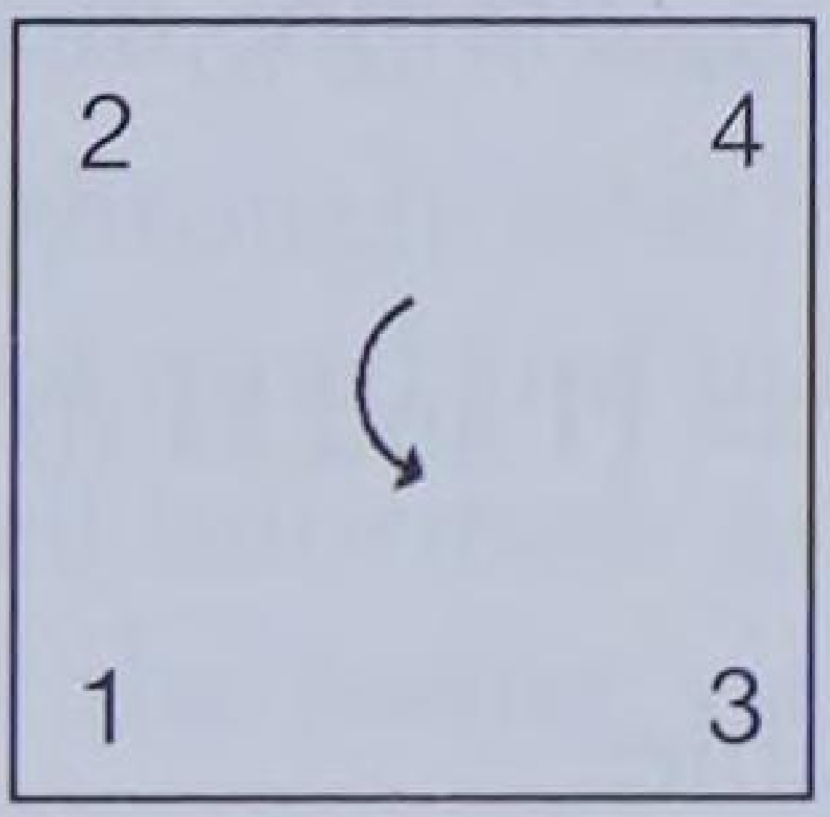

Figure 11.2

For lack of a better name, call this quarter-turn move **Q**. Notice the uniform rotating 'shift' in the numbered corners. The corner designated by 1, which in the original image is located upper-left, when applying **Q**, moves now to occupy the lower-left corner, and the other numbers follow—2 now occupies the upper-left, 4 now occupies upper-right, and so on.

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[1] I am borrowing almost completely from Yarnelle (1964), pp. 42–46.

We can introduce now the half-turn:

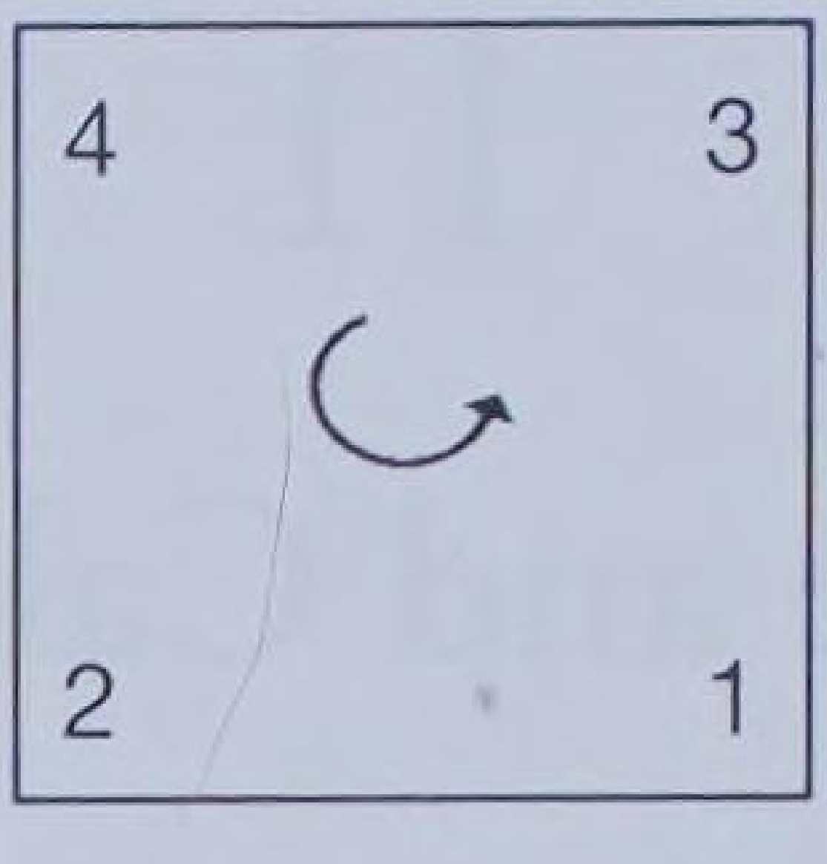

Figure 11.3

Starting with our original image, and rotating a half-turn counter-clockwise (180 degrees), the result will look like the above. This is motion **H**. When applying **H**, 1 moves to now occupy the lower-right corner, 2 the lower-left, 4 the upper-left—the numbered corners moving together. We could now go on to introduce the three-quarter turn **T**, a 270 degree turn (no need to provide the image at this point). And, of course, a full 360 degree turn, which I will call **E**. Clearly if we rotated the square 360 degrees around its centre, we would end up with exactly the same image as the one with which we started. (Notice that moving the square by way of motion **E** would be the equivalent, at least with respect to the way it looks, of having done nothing at all to the square.)

It is easy to see that if we turned the original square a quarter-turn, and then again a quarter-turn, we would have moved the square around in such a way that the resulting image would look identical to the image produced by simply moving the original square a half-turn. Or if we moved the square a half-turn, and then again a half-turn, the resulting image would be identical to the image produced by moving the square a full-turn (or not at all).

Let ⊕ be the operator we use to combine moves. Given what we just noted, it is easy to see that:

$$\mathrm{Q} \oplus \mathrm{Q} = \mathrm{H} \text{ and } \mathrm{H} \oplus \mathrm{H} = \mathrm{E}.$$

We could make the following table:

Table 11.1

⊕	Q	H	T	E
Q	H	T	E	Q
H	T	E	Q	H
T	E	Q	H	T
E	Q	H	T	E

We could go further and define other moves. Treating the square as though it were a piece of paper (as a three-dimensional object), we could move it in ways, adding to **Q**, **H**, **T**, and **E**. For example, we could take the square piece of paper and 'flip' it around an axis drawn horizontally through its centre. Another move will 'flip' it around an axis drawn vertically through its centre. And, yet other moves might 'flip' it around an axis drawn diagonally, corner to corner. Since I can make my point with the four earlier moves—**Q**, **H**, **T**, and **E**—I will not further discuss the details of these other 'axis' moves.

Return, then, to the first set of four moves (**Q**, **H**, **T**, and **E**). We can see by looking at the table (see Table 11.1) that the set is closed with respect to the binary operator $\oplus$. If we went further, we would find that this operator is associative. **E** will be the set's identity element. And, the set will include inverses for each element. For instance, since $\mathbf{Q} \oplus \mathbf{T} = \mathbf{E}$, and **E** is the identity element, **T** is **Q**'s inverse (and, as it turns out, **Q** is **T**'s). Since the above meets the four conditions of a group (even when including the additional moves), this set of moves M, the binary operator $\oplus$, and the identity element **E** form a group (M, $\oplus$, **E**).

Notice that with the two-dimensional square, and later the three-dimensional 'square' (where we expanded our example and took the square to be a square-shaped piece of paper), their structures are closed *mathematical* structures, where by 'closed' it is meant that the system is cyclical. The square piece of paper is nothing short of a fully informed mathematical system, where even simply moving it around its centre and axes (horizontal, vertical, and diagonal), it (in meeting the conditions of a group) exhibits all the conditions necessary (and sufficient) for a mathematical reality. Recall that, early in his career, Leibniz took corporeal substance or body to be the *union* of form (cast as mind) and matter.[2] And so, unlike matter *simpliciter*, which serves as the medium for the actualization of structure (form)—or as it was put, matter was that metaphysical medium by way of which, or through which, mind acts—*bodies* or *material objects*, which arise as the union of mind and matter, are found on the ontological scene already mathematically well informed, as our square or piece of paper.

We need not be confined to the earlier view, of course. In focusing on his later view, Leibniz seems to have thought of extension (or the plane) as the relational nexus arising from the monads, the monads taken here to be analogous to Euclidean points. Space or extension, in other words, is a set of relations holding between the monads. The relations are uniform, and as De Risi puts it:

> Such uniformity can be more or less understood as what we would today call homogeneity or, perhaps better, isotropy of space. It is the property that grants the transportation of figures through congruence, and thus free motion.[3]

In his later view, then, the relational nexus seems to be Leibniz's way of conceiving Cartesian extension. Although we need to be careful here, we could think of it as the medium through which shapes (bodies produced by modal division) will move (understood here in terms of transformations). A shape will be determined by a function or equation (its notion), and its movement *as a single unified object* will be determined by a system of equations (solvable by way of determinants). Understood this way, and in line with the earlier view, *bodies* or *material objects* are found on the ontological scene already mathematically well informed, again as our piece of paper (or, recalling a passage cited earlier, as the sheep's body).

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[2] Mercer and Sleigh (1995), pp. 67–123; and Mercer (2001).

[3] Vincenzo De Risi (2005), p. 130.

Recall that Descartes took mathematics to deal with modes of extension. Geometry, he tells Gassendi, deals with the boundaries or shapes of bodies. In terms of what we are developing here, we might rephrase this and say that geometry deals with mathematical structures, such as squares (or cubes), which divide the plenum—it does not deal with the plenum *sans* modal division.[4] This does not diminish the significance of extension (*sans* division), of course, for extension serves as the medium by way of which the division is made actual. It is the *subject* of division. As James Collins puts it, the nature of matter for Descartes must be capable of receiving 'the pattern of movements regulated by the laws of mechanics'.[5] And, as we entertained earlier, if Descartes holds a view anything like that of Aquinas, matter (or extension) is taken to be the *only* substrate that, when divided, will yield a mathematical reality (it is that substrate that, when divided, is only rationally or conceptually distinct from *quantity*).

This idea of not rooting mathematics in the immaterial substance (to mind), and especially of not rooting it directly in God, is not limited to just Descartes (or Aquinas). Leibniz, for instance, appears to hold something similar. In the *Discourse*, for example, he notes that number is not a perfection (it does not emit of a highest degree), and so is not attributable to God.[6] Shape also fails the criterion for being a perfection.[7] Thus, along the lines of Aquinas' view, it would seem that in a world in which there exists *only* the divine being—that is, only infinite mind—mathematics (arithmetic and geometry specifically) is not something that arises. In Part IV, I will address the philosophical worry over whether one could conceive (and hence *perform*) mathematics even though extension did not exist. That is, I will consider the concern over whether there is a difference (for Descartes specifically) between one's thinking *about* extension and extension as something existing independently of thought. So, I will set that concern aside for the moment and move on to talk about a connection between machines and inherent mathematical structure.

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[4] D. M. Armstrong develops an idea similar to this in his *A Combinatorial Theory of Possibility* (Cambridge: Cambridge University Press, 1989). There, in the section on mathematics, Armstrong works out how the conditions for mathematics are established in a world where properties are taken to 'divide their extension' (pp. 127–31). This is a phrase that he wants to echo Quine's semantic notion, expressed by the phrase 'divides its reference' (p. 127). Here, the idea Armstrong emphasizes—mutually exclusive properties understood to collect and divide objects by sorting them into their respective extensions (classes)—echoes what I have set down as the result of Descartes's method of analysis. Even so, Armstrong's phrase can also be taken to echo what I am literally calling 'dividing extension', where the division in extension constitutes the production of quantity or number, something that Armstrong seems to be suggesting in his section, 'The Nature of Number' (pp. 126–33).

[5] James Collins, 'Descartes' Philosophy of Nature', in *American Philosophical Quarterly*, monograph series (monograph no. 5), Nicholas Rescher (ed.) (Oxford: Basil Blackwell, 1971), p. 22.

[6] *Discourse*, Section 1. [7] Ibid.

As has been mentioned several times throughout the book, one of the earliest combinatorial systems developed that had an impact on Descartes and Leibniz was Raymond Lull's *Ars Magna* and *Ars Brevis*. According to J. N. Hillgarth, Lull set up his combinatorial system by establishing what he took to be nine basic categories, the *Lullian Dignities* (a Dignity being something like a Platonic Idea), from which all other categories of the system were to be synthesized.[8] Ultimately, in knowing how things were synthesized, we would in turn know how they are reduced to the Dignities, or atomic categories. In other words, synthesis would yield analysis of complex entities. However, as Leibniz complains, Lull requires that the Dignities, the atomic categories, be given up front. That is, Lull's system was not set up so that its application would *reveal* the atomic categories. This is where Descartes's method of analysis had the advantage.

Lull constructed a combinatorial device, a 'wheel', which turned around its centre. Consider Wheel I:

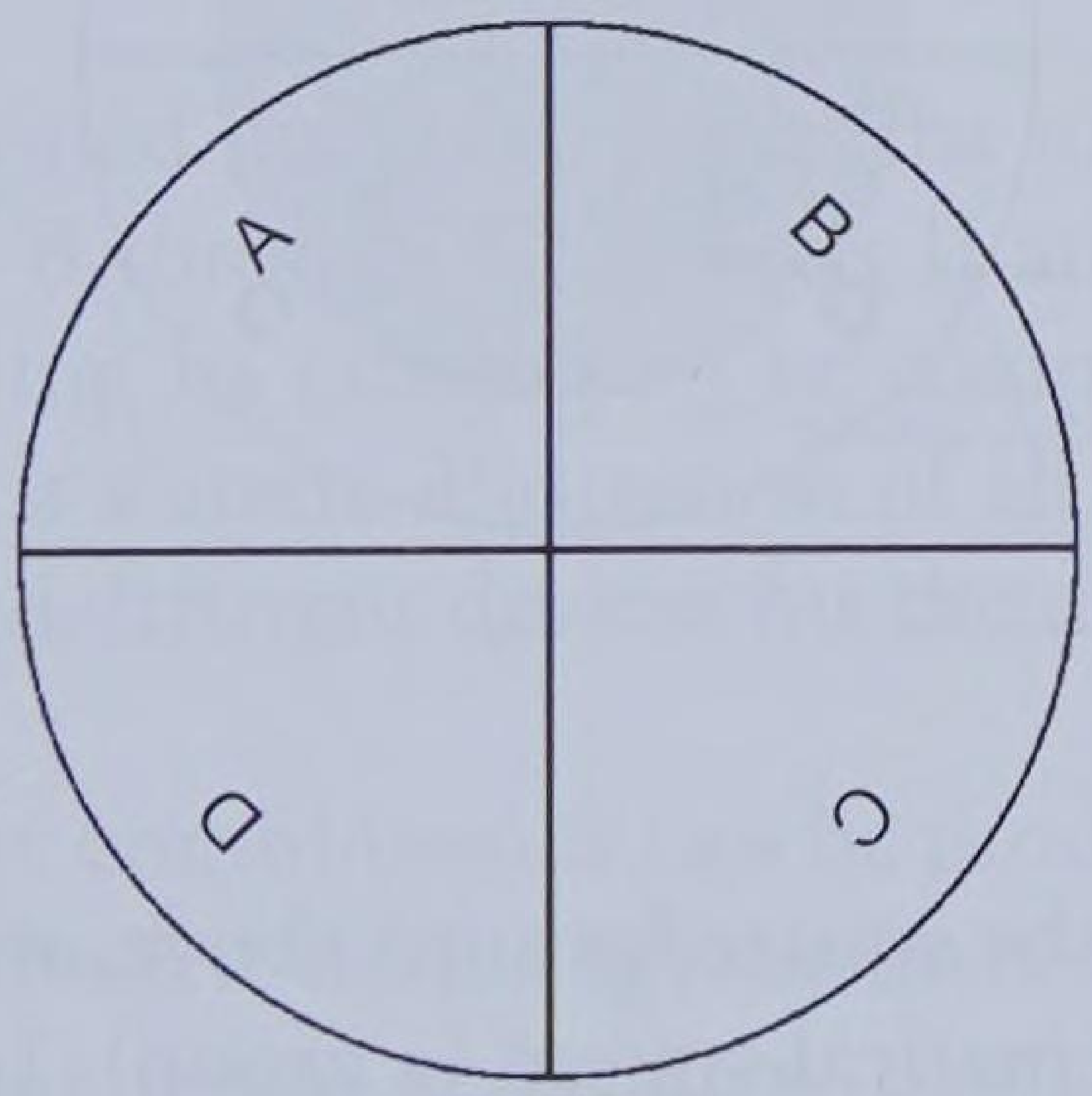

Figure 11.4

For the sake of simplicity, I have divided the wheel (in Figure 11.4) into only four parts, A, B, C, D, which represent four basic categories. Now, let's add to Wheel I a second, Wheel II.

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[8] J. N. Hillgarth. *Ramon Lull and Lullism in Fourteenth-Century France* (Oxford: The Clarendon Press, 1971), p. 10.

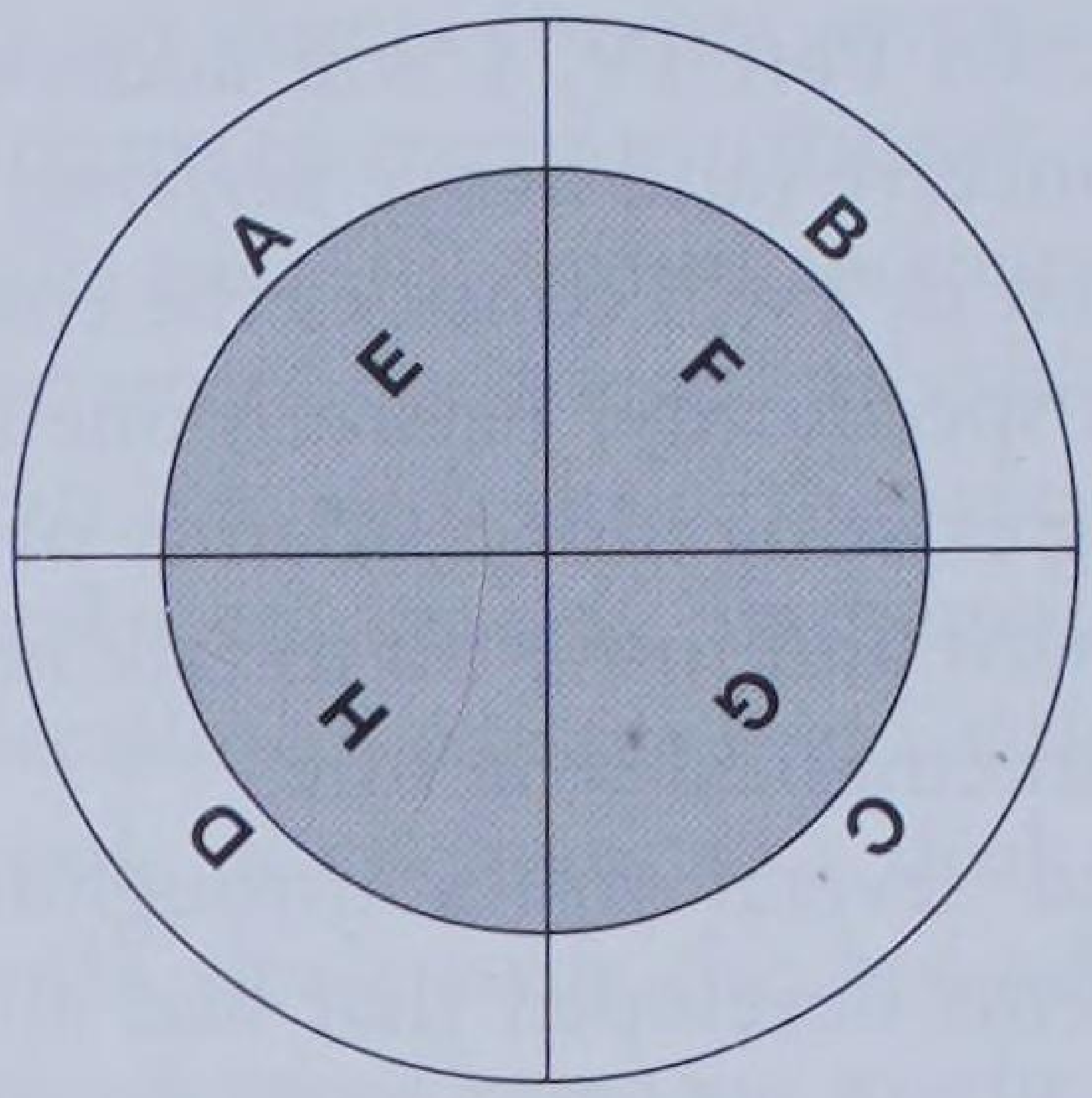

Figure 11.5

Wheel II sits directly on top of Wheel I, where Wheel II has also been divided into four parts, E, F, G, H, which also represent four basic categories. In their current position, the Wheels give us the following combinations:

AE, BF, CG, DH.

Let's turn Wheel II a quarter-turn clockwise:

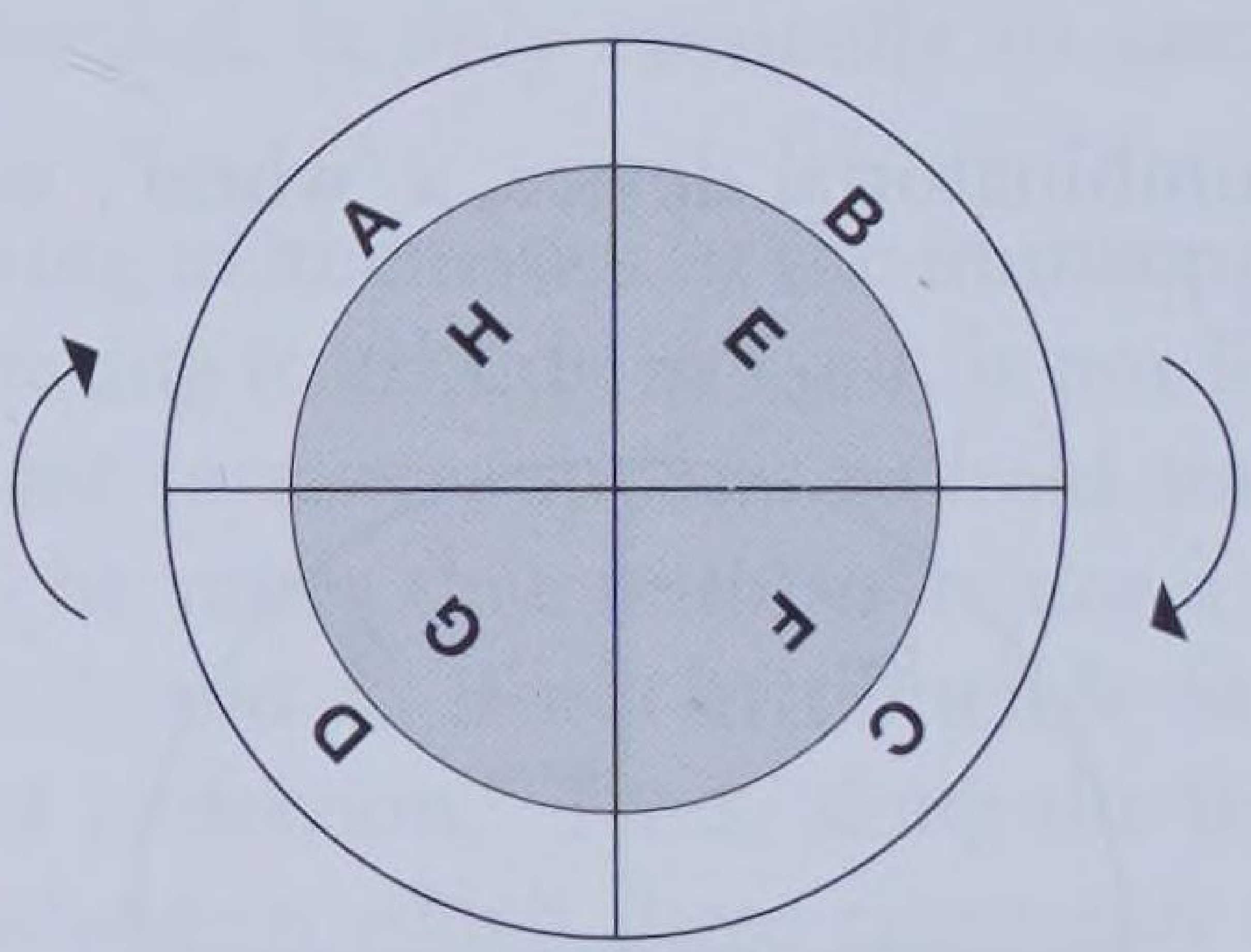

Figure 11.6

From our examination of the square (group) above, we know that the move here is a constituent of a mathematical system (a group). In connection to *this* move, the Wheels now give us the following combinations:

AH, BE, CF, DG,

where, as we will see shortly, Euler, Cayley, and others showed that the *combinations* are constituents of a mathematical system (which is mathematically—*necessarily*—internally-related to the system expressed by the moves). Below is an image of a device taken from an early seventeenth-century printing of Lull's book:[9]

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[9] The image comes from Frances Yates' *Lull & Bruno*, vol. 1 (London: Routledge and Kegan Paul, 1982), p. 117.

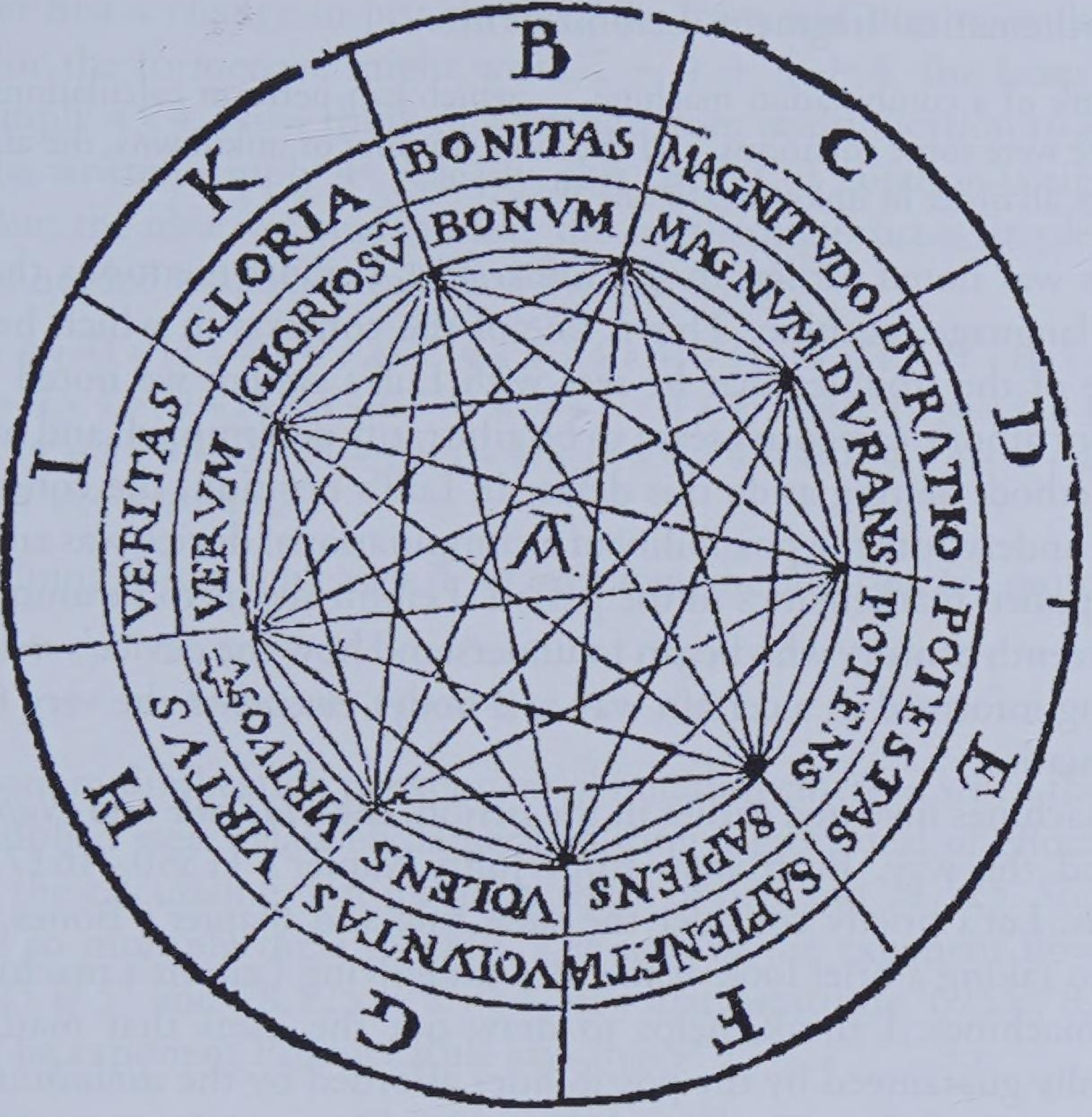
B
C
D
E
F
G
H
I
K
BONITAS
MAGNITVDO
DVRATIO
POTESTAS
SAPIENTIA
VOLVNTAS
VIRTVS
VERITAS
GLORIA
BONVM
MAGNVM
DVRANS
POTENS
SAPIENS
VOLENS
VIRTVOSV
VERVM
GLORIOSV
A

Figure 11.7

Notice that Wheel I is divided into nine parts (the nine Dignities), represented by letters of the alphabet, B through K. Wheels II and III are also divided into nine parts, each part naming its category. The complex geometrical pattern in the device's centre region is a static illustration of all the various combinations. Lull apparently constructed different devices for the different sciences, including theology.

It is easy to see that more combinations can be produced with a three-wheeled device than with a two-wheeled device (given each wheel is only divided into nine parts, and we do not count redundant combinations). Lull apparently constructed a fourteen-wheel device. The idea seems to have been that the device could produce, by way of combining the fourteen wheels, entire propositions consistent with both Dogmatic and Apologetic Christian texts. An algorithm followed by the device's user seems to have allowed the user to produce the propositions (which may have been *answers* to theological questions). If so, it was nothing short of a primitive computer.

In a mathematical fragment, Leibniz writes:

> I often think of a combination machine . . . which can perform calculations on letters. And if there were some equations, and the same number of unknowns, the aim would be to eliminate all order of unknown to one.[10]

And, as was noted earlier, in the *Dissertatio*, Leibniz mentions the idea of a universal language machine. This is one of the contexts in which he mentions Lull. One of the troubles that he sees with Lull's system we noted is that the Dignities or atomic categories seem to be arbitrarily determined, and not revealed by any method. Setting aside this defect in Lull's position, the constellation of *ideas* that underwrote the possibility of a computational device was emerging as a topic of applied mathematics in the period. Leibniz seems to be among those in the seventeenth century who began to understand how the device's *structure* (that is, its being informed in a certain way as a body) *embodied* the very foundation of mathematics.

Two machines invented earlier in the period, the *slide rule* and *Napier's Bones*, would lead the way. Both stem from John Napier's (1550–1617) work on logarithms. Let's briefly consider the slide rule and Napier's Bones, and work our way to taking a brief look at the idea underlying Leibniz's machine. A look at these machines, I think, helps to draw out the ideas that mathematics is conceptually guaranteed by the possibilities afforded by the *divisibility of matter* (giving rise to 'body') and by the subsequent *combinatorial nature of body*.

Napier's invention of the logarithm was rooted in his wanting to make calculating with very large numbers manageable. Such calculating would have been necessary aboard ship, for example, when working out crucial navigational data. The astronomer Tycho Brahe was among those who made it clear to Napier in 1594 how the pains of tedious and time-consuming calculations hampered astronomical work.[11] Napier subsequently communicated a summary of his work on logarithms to Brahe.[12] Logarithms are basically ways of dealing more easily with exponents. Exponents were invented as a way of dealing more easily with multiplication of large numbers. And, multiplication became the operation of choice when dealing with the adding of numbers multiple times. So, the history (of practice) here could be viewed as moving from regular addition to regular multiplication, from regular multiplication to exponents, from exponents to logarithms, where each step in this history stems from the desire to make calculating easier.

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[10] Gottfried Leibniz, *Die Mathematical Schriften von Gottfried Wilhelm Leibniz*, vol. 3, von C. I. Gerhardt (New York: George Olms Verlag, 1978), p. 572.

[11] J. E. Thompson, *A Manual of the Slide Rule: Its History, Principle, and Operation* (New York: D. Van Nostrand Company, Inc., 1930), p. 2.

[12] Ibid.

Consider first a change in just the *notation* from addition to multiplication. Whereas for the former we might write $4 + 4 + 4 + 4$, the latter allows us to write simply 4×4. Now think of a change from multiplication to exponents, 4×4 can be written simply 4^2. Clearly, this last bit of notation is simpler than the first. But, the idea was also to make the calculating process simpler. So, let's say that we needed to quickly calculate the following:

$$(4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4)$$

We make doing this much easier if we exchange the addition for multiplication. So:

$$(4 \times 4) \times (4 \times 4)$$

Moving from multiplication to exponential notation emerged when the numbers to be multiplied were relatively large, requiring a great deal of effort from the one doing the calculating. For instance, think of 625 and 78,125. Suppose that we needed to multiply these quickly. The idea of the exponent now becomes helpful. 625 is 5^4 and 78,125 is 5^7. So, let's first substitute '$625\times78{,}125$' with '$5^4\times5^7$'. The exponent Product Rule says that $x^m x^n = x^{m+n}$. So:

$$5^4 \times 5^7 = 5^{4+7} = 5^{11}$$

Now, if we had a table that simply listed all the numbers associated with their respective exponential equivalents—for example, 5^0 will be correlated with 1, 5^1 with 5, 5^2 with 25, . . . , 5^7 with 78,125, and so on—then all we would have to do is look to the entry of 5^{11} and record the number correlated with it (which will be a very large number, of course). Notice here that the move to exponents was a move from multiplication back to some form of addition. That is, '$625\times78{,}125$' was recast as simply '$4 + 7$' (where 4 and 7 were exponents of base 5).

An early procedure for converting multiplication of large numbers back to some manageable form of addition was called *prosthaphaeresis*, developed in the sixteenth century, primarily within the mathematical field of spherical trigonometry.[13] Napier, in his early work on using logarithms for converting multiplication to addition, works out the logarithms for sines and tangents of angles, calculated to seven decimal places. The first of two works, *Mirifici logarithmorum canonis descriptio*, was published in 1614. Though it is unclear, given the emphasis on sine and tangent it would appear that the trigonometric-based *prosthaphaeresis* had some influence on his thinking.

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[13] See Clavius and Pitiscus, 'On Prosthaphaeresis' (a fragment) in Smith (1929), pp. 455–8, and Clavius, 'On Prosthaphaeresis as Applied to Trigonometry', in Smith (1929), pp. 459–62.

In the second work, *Mirifici logarithmorum canonis constructio* (1619), Napier writes:

> A logarithm table is a small table by the use of which we can obtain a knowledge of all geometrical dimensions and motions in space, by a very easy calculation . . . It is picked out from numbers progressing in continuous proportion.[14]

He recognizes two kinds of continuous progressions:

> Of continuous progressions, an arithmetical is one which proceeds by equal intervals; a geometrical, one which advances by unequal and proportionally increasing or decreasing intervals.[15]

Although not perfectly clear at first, his idea seems to be that there is a way to 'map' (by way of a function) from an arithmetical progression to a geometrical one, and vice versa.

The import of this becomes significant when we reconsider the numbers 625 and 78,125, which are 5^4 and 5^7 respectively. Again, supposing that we are asked to multiply 78,125 by 625 (which, recall, is the shortcut to adding 78,125 in repetition 625 times), Napier's insight becomes valuable, for as we did earlier, we could translate these larger numbers to exponential form, and then calculate $5^4 \times 5^7$, which is 5^{4+7}, or 5^{11}. I should mention that Napier did not originally construe logarithms in terms of exponents. Henry Briggs, a professor of mathematics at Oxford, and correspondent of Napier's, introduced the idea (in 1617) of understanding logarithms in terms of the now familiar base 10.[16] Be that as it may, with reference to exponents I can make Napier's *idea* clearer. It should also be added that he would suggest that we provide tables that would allow the one calculating to, in effect, look at the entry for 5^{11}, say, which would give him or her the translation *back* to the original progression of which the larger numbers are members. In providing such a table, *someone* had to do the long calculation originally, of course, but it needed to be done only once, which saved future calculators from having to do it. They simply had to look to the table.

Here is Napier's idea. In *Mirifici logarithmorum canonis constructio*, he sets down two parallel lines, TS and BI, one above the other.[17]

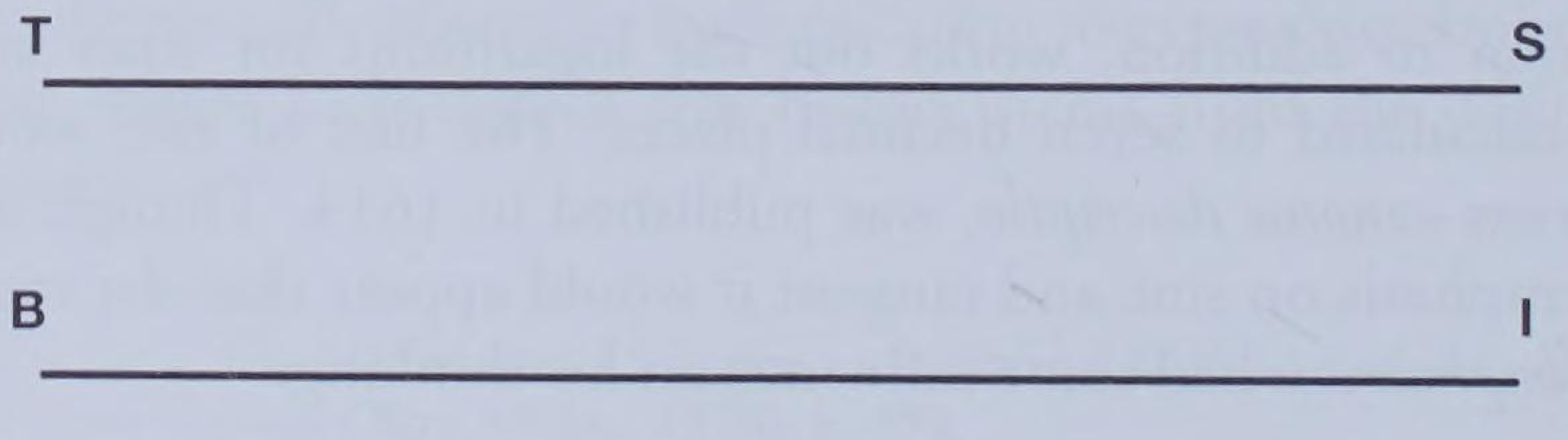

Figure 11.8

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[14] John Napier. *Mirifici logarithmorum canonis constructio* (1619), translated by W. D. Cairns, in Smith (1929), p. 149. I shall refer to this as 'Napier (Smith 1929)'.

[15] Napier (Smith 1929), 149.

[16] Ibid., p. 3.

[17] Ibid., p. 152.

Now, suppose that there is a point *g* on TS, which starts at T, and moves with a uniform velocity towards S, stopping at D. Also suppose that there is another point *a* on BI, which starts at B, and moves with a determined acceleration, the acceleration in some proportion to the velocity of *g* on TS, towards I, stopping at C, at the very moment *g* stops at D. This sounds complicated, but it is not. It looks as follows:

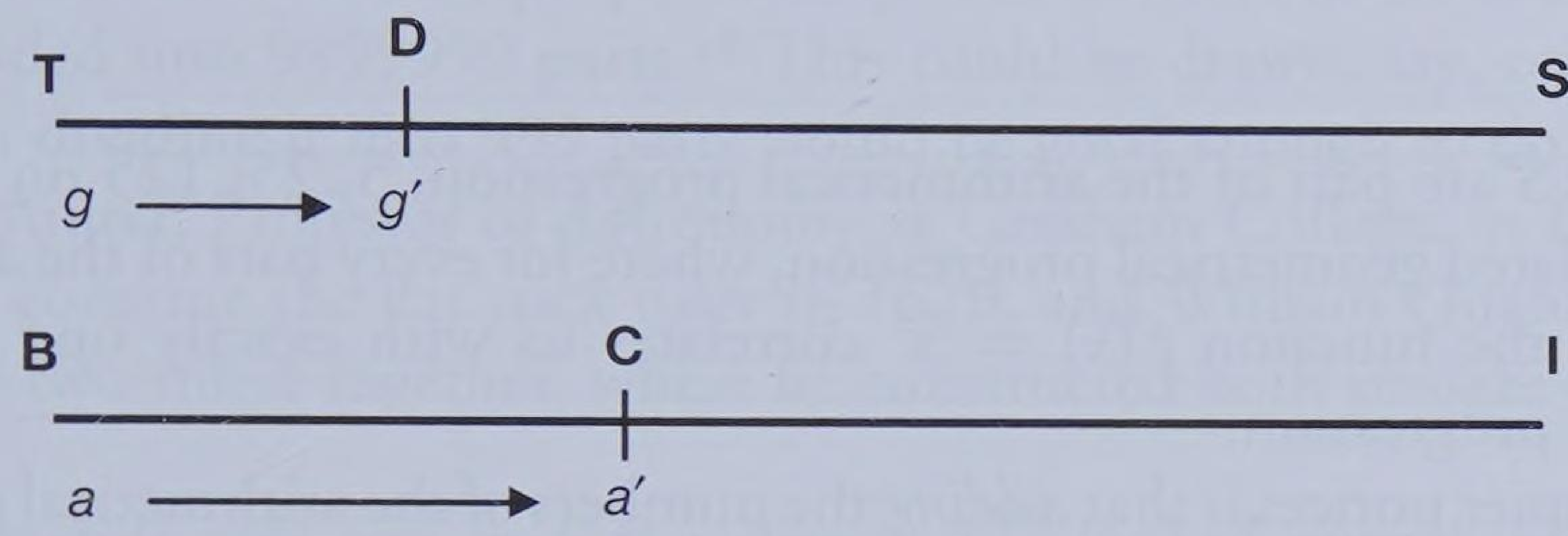

Figure 11.9

In the time it took *g* to move from T to D (when *g* arrives at D, *g* is now *g*′), *a* was able to move from B to C (when *a* arrives at C, *a* is now *a*′). There is a function that can map us from *g*′ at D to *a*′ at C. Clearly, the magnitude of line segment BC will be proportionately greater than that of line segment TD. Where *g*′ is allowed to move again, from D this time, the same distance as before, our function will determine that *a*′ will move proportionately further from C some distance towards I. Thus:

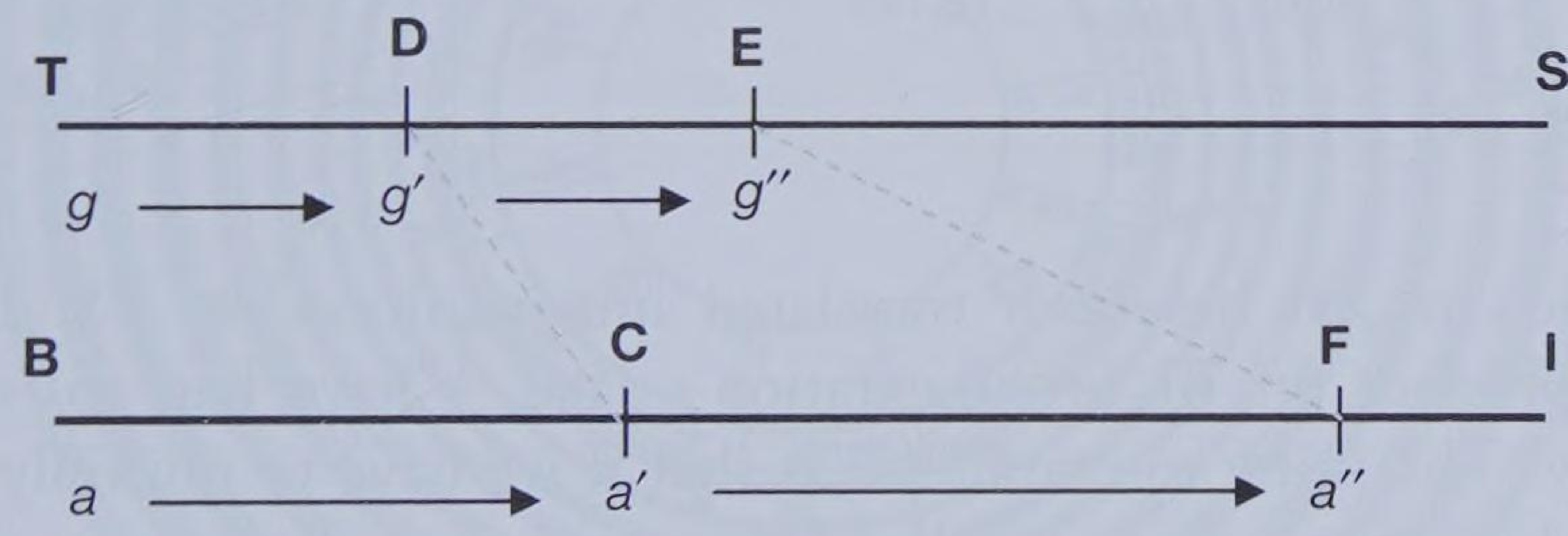

Figure 11.10

The progression on TS is arithmetical whereas the progression on BI is geometrical. This means that the intervals on TS are continuous and equal, whereas the intervals on BI are continuous but not equal—they increase proportionately. An example of the sort of function that would map from TS to BI is the exponential function—the form of which is $f(x) = b^x$, where b is a fixed base and x is the exponent. So, let the base be 5. The function is $f(x) = 5^x$. Where '$f(x)$' is just another way of expressing 'y': when $x = 0$, $y = 1$; when $x = 1$, $y = 5$; when $x = 2$, $y = 25$; when $x = 3$, $y = 125$, and so on. Now, let the exponent (the x value) be located on TS and the correlated y value on BI:

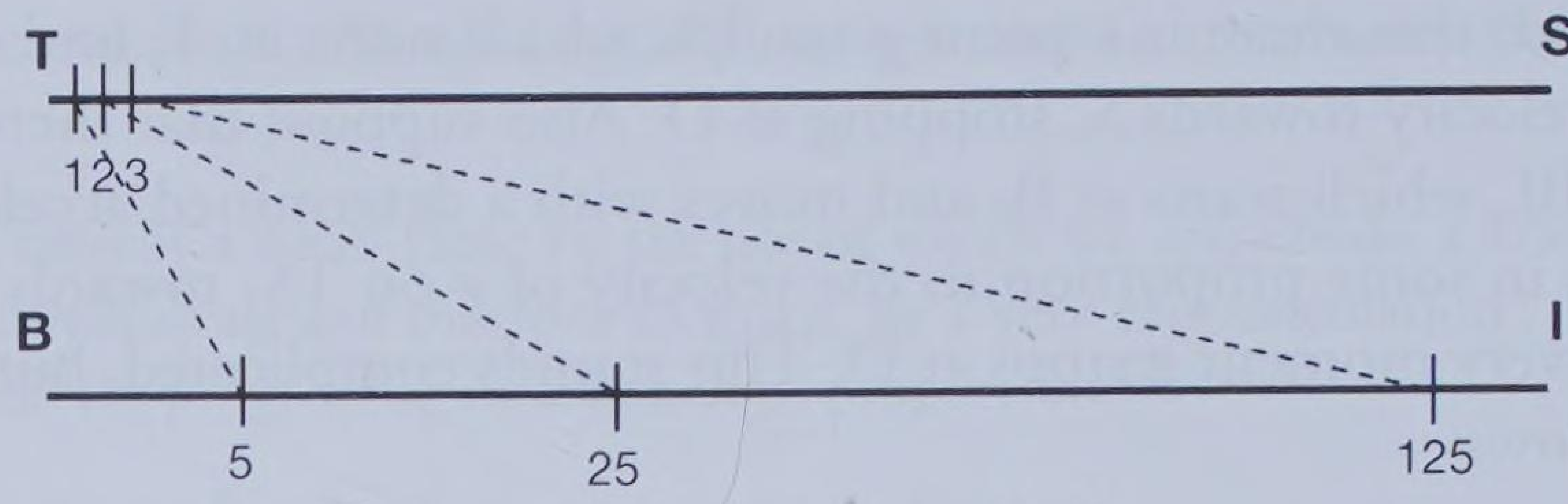

Figure 11.11

1, 2, 3 on TS are part of the arithmetical progression, 5, 25, 125 on BI are part of the correlated geometrical progression, where for every part of the arithmetical progression the function $f(x) = 5^x$ correlates us with exactly one part of the geometrical progression.

What Napier notices is that *adding* the numbers of the arithmetical progression is the same as *multiplying* the numbers of the geometrical progression. So, thinking now of the example of 78,125 multiplied by 625, where we know that these are 5^7 and 5^4 respectively, instead of multiplying 78,125 by 625, we can simply add 7 and 4 on TS:

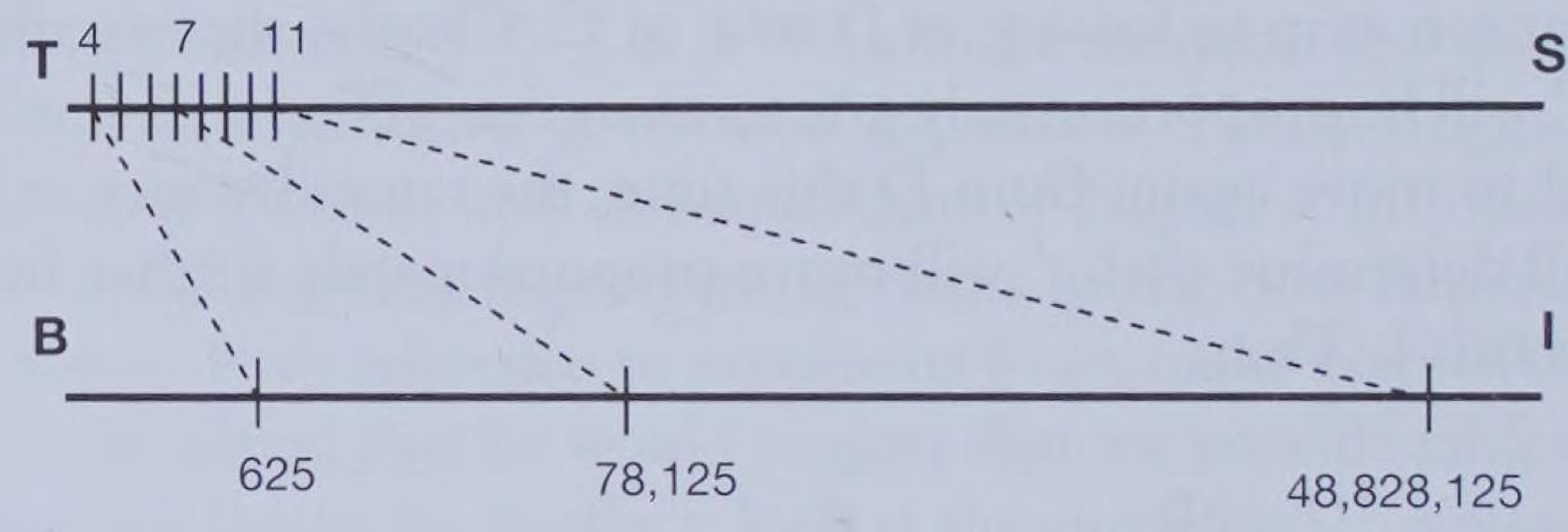

Figure 11.12

Multiplication on BI has been translated into addition on TS. In order to locate the product on BI for illustration's sake, I have had to 'squeeze' BI considerably. The thing to emphasize is that if we have to multiply something like 78,125 by 625, we can instead add their arithmetical counterparts on TS, 7 and 4.

Logarithms are exponents. We know that $4^2 = 16$. Since the logarithm is the exponent, which in this case is 2 (the base is 4) the notation is as follows:

$$\log_4 16 = 2.$$

This is read: 'The log of 16 base 4 is 2.' The log of 64 base 8 is 2, or $\log_8 64 = 2$, which is just another way of noting $8^2 = 64$; or thinking about the above, the log of 78,125 base 5 is 7, or $\log_5 78{,}125 = 7$, is another way of noting $5^7 = 78{,}125$. The general formula, then, is:

$$\log_b x = y, \text{ which is another way of noting } b^y = x.$$

In thinking about the above case in which 78,125 was multiplied by 625, clearly if we wanted to draw TS and BI, drawing the latter would take a bit of doing (and may be physically impossible if the numbers become too large). BI will be just too long. Napier's solution to this, in knowing that the interval of a geometrical progression can also *decrease* proportionally, was to think of BI as a single unit, ideally divided into 9999999 parts.[18] This could be drawn, say, on a flat stick, where even dividing it into 999 parts would be good enough to do some work. Edmund Gunter, Professor of Astronomy at Gresham College, in London, was the first to construe the flat stick ruler in 1620, and William Oughtred was the first to put two rulers together, where he constructed both straight and circular rulers.[19]

The first slide rule was in fact a device that looked remarkably like Lull's wheels.

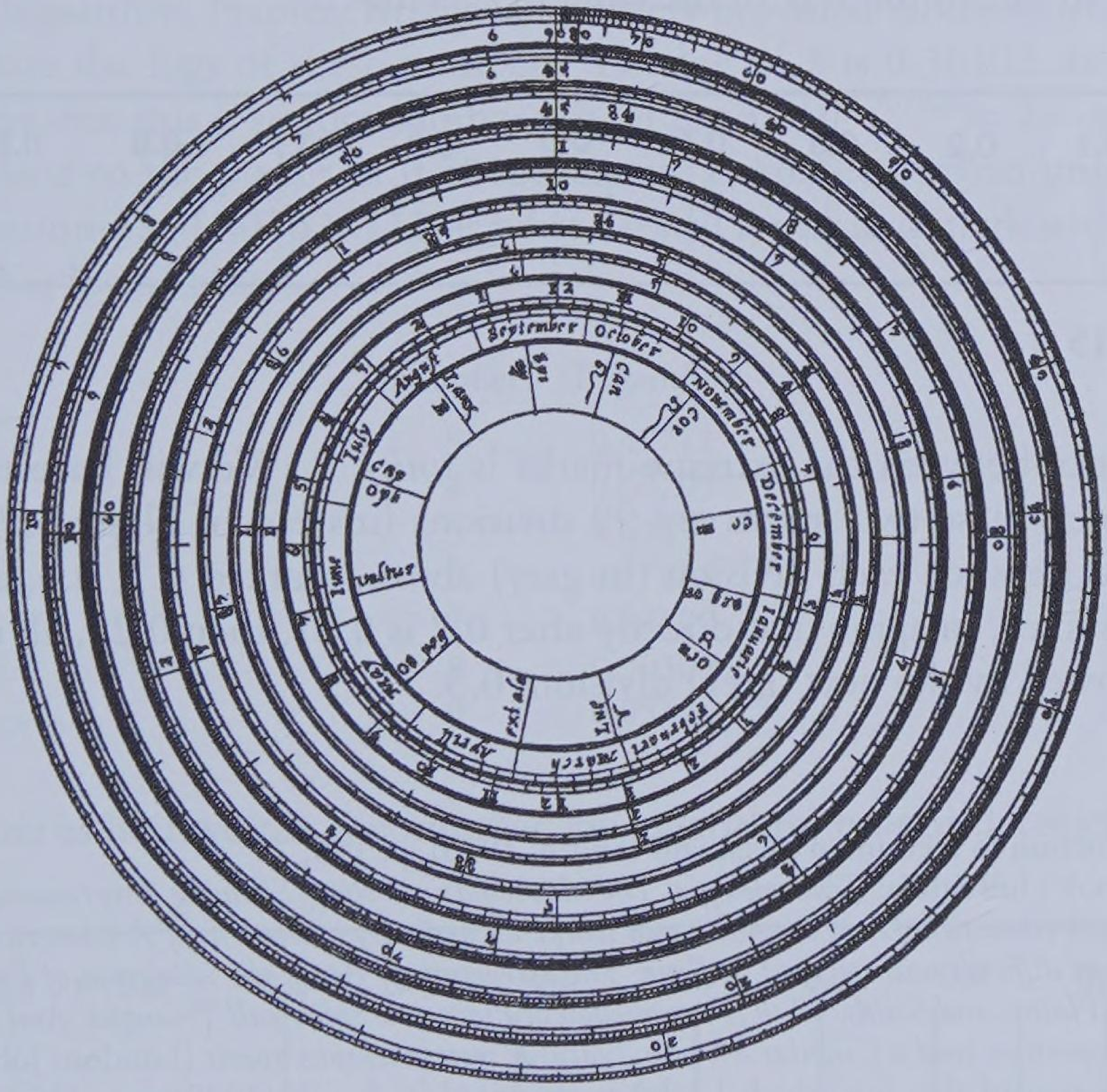

Figure 11.13

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[18] Napier, *Constructio* (1619), in (Smith 1929), pp. 150–1.

[19] Thompson (1930), pp. 4–5.

William Oughtred's book, *Circles of Proportion* (1632),[20] discusses the device, and images are found there, and in the second and third printings of Richard Delamain's *Grammelogia* (1633).[21] The above illustration (Figure 11.13) comes from Oughtred's book.[22]

We can see how the slide rule works by building it. First, let's start with a flat elongated stick (instead of a disk). Divide this stick into 10 equal parts (small margins on both ends).

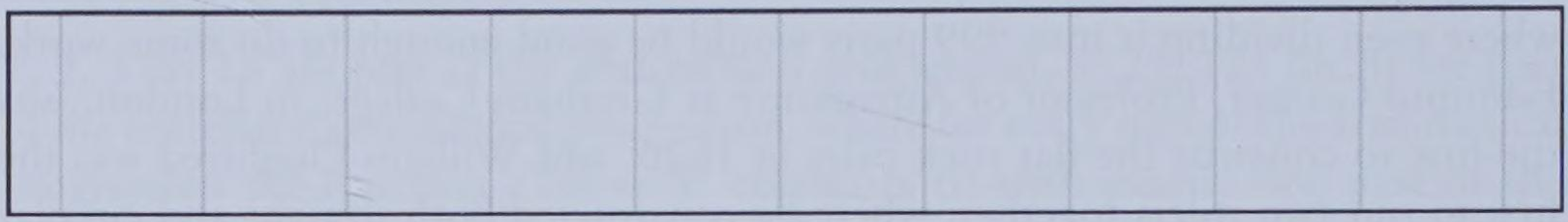

Figure 11.14

Label the first and last marks '1' and the interim marks in the order '0.1', '0.2', '0.3', and so on, all the way to the end, '0.9'. Thus:

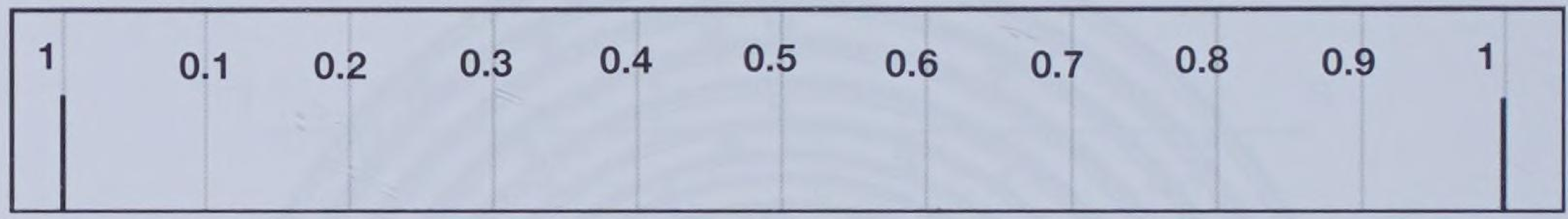

Figure 11.15

The distance between the extreme marks is *unity*, 1. We will have to imagine that between these two marks are 99 divisions (instead of Napier's 9999999), 9 divisions between each division (in grey) above, marked 0.2, 0.3, and so on. So, for instance, imagine that directly after 0.2 is 0.21, then 0.22, all the way to 0.29, followed by the next (grey) division, 0.3.

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[20] Introduction to section on Delamain (Smith, 1929), p. 156.

[21] The book's full title is: *Grammelogia, The Mathematicall Ring. Shewing (any reasonable Capacity that hath not Arithmeticke) how to resolve and worke all ordinary operations of Arithmeticke. And those which are most difficult with greatest facilitie: The extraction of Roots, the valuation of Leases, etc. The measuring of Plaines and Solids. With the resolution of Plain end Sphericall Triangles. And that onlely by an Ocular Inspection, and a Circular Motion. Naturae secteta tempus aperit* (London: John Haviland, 1630). A portion of this (on which I rely) is reprinted in Smith (1929), pp, 156–9, edited by Florian Cajori.

[22] The image, I believe, is reproduced from (Smith, 1929), p. 161. Even so, I obtained a digital image of the sliderule at http://web.mat.bham.ac.uk/C.J.Sangwin/Sliderules.

Next, we calculate the logarithms for 2, 3, 4, up to 9. Here, the logarithms will be base 10. It is important to keep in mind that the logarithm function is an inverse of an exponential function. So, the intervals of the geometrical progression will decrease proportionally, instead of increase. As in the above case, the exponents will constitute the arithmetical progression. So, let's begin marking our ruler. For a start, then, what we want to know is the exponent (or power) that 10 must be raised to in order to equal 2. That is, what we want to solve is x, for the equation:

$$\mathrm{Log}_{10}2 = x \text{ (or, in exponential notation, } 10^x = 2\text{)}.$$

Base 10 logarithms are called *common logarithms*, and are usually not written with 10 in the place of base. And, as I noted earlier, it would not be until Henry Briggs, in his book on logarithms following soon after Napier's, that the logarithm base 10 would be instituted as the common logarithm. The above is usually written $\log 2 = x$, which is to be read, 'the log of 2 is x', where x is said to be the logarithm. Napier, Briggs, and others provided tables so that we need not calculate the logs of these numbers. The log of 2 is 0.30102. It should be crystal clear that this is simply another way of saying $10^{0.30102} = 2$—which is to say, 10 raised to the power of 0.30102 equals 2. Somewhere on unity (among the 99 divisions) is 0.30102. This is what we will locate and mark with '2'. Here are the other logarithms:

Number	Logarithm
3	0.47712
4	0.60205
5	0.69897
6	0.77815
7	0.84509
8	0.90308
9	0.95424

Now, locate as best we can the number (by logarithm) on our flat stick.

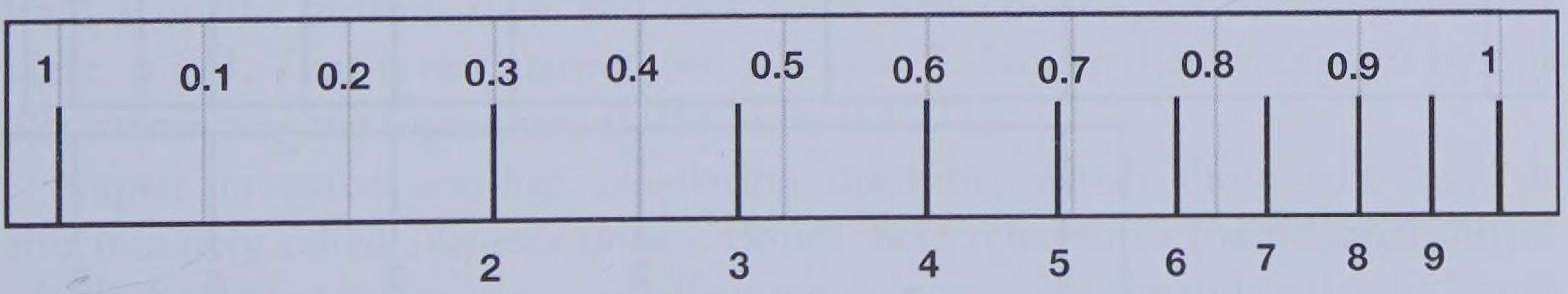

Figure 11.16

Since the log of 2 is 0.30102, and we are imagining unity divided up into only 99 parts, we will be restricted to two decimal places. So, 0.30102 will be rounded up to 0.30 (we will do our best to 'eye' it here), marking it with '2'. The log of 3 is 0.48, so we find its location, marking it with '3'. What we are doing, of course, is mapping our arithmetic progression (the exponents 0.30102, 0.47712, to 0.95424) onto a geometric progression (here 2, 3, to 9), the former are the logarithms of the latter. So, instead of the intervals increasing, as in the earlier case, the intervals are *decreasing* proportionally, and the series is approaching (or as Wallis and Newton put it, *converging* on) the terminal division on our ruler (marked by **1**) to the far right.

We will go ahead and move to the next step, marking on the stick the numbers 2, 3, 4, and so on, (once we do this, we can ignore the original divisions of 0.2 to 0.9). Now, let's make a second stick, where the markings mirror those of the first stick, and place the first stick on top, the second on bottom:

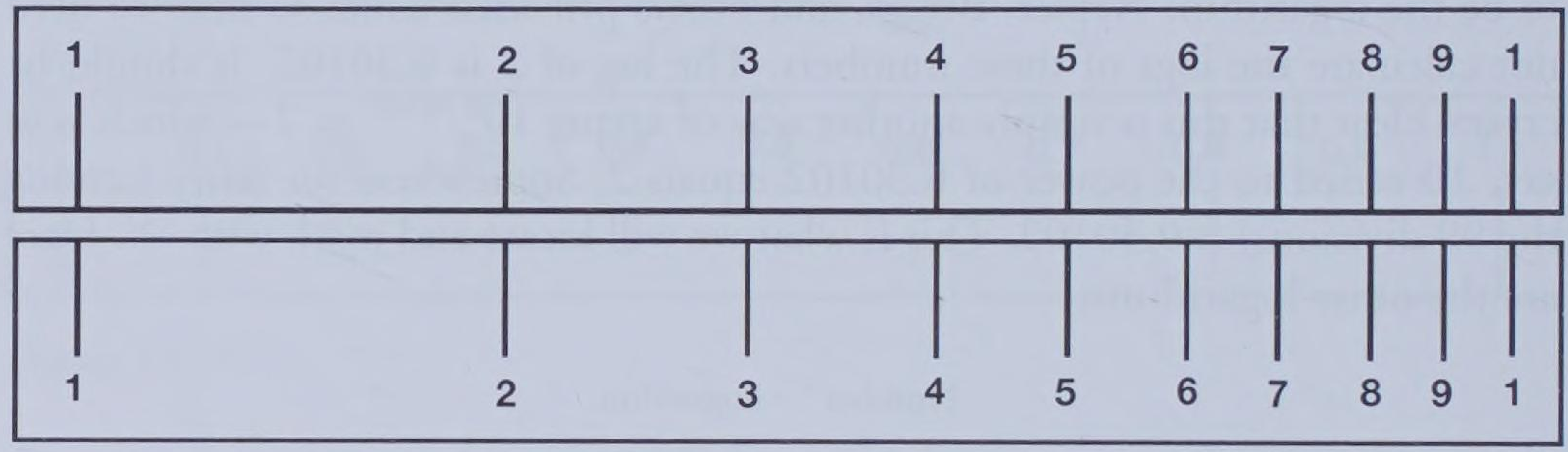

Figure 11.17

This is our slide rule. Multiplication and division are easily done with this machine. Suppose that we want to multiply 2×3. We simply 'slide' the bottom stick (ruler) to the right, lining up the bottom ruler's **1** with the top ruler's 2:

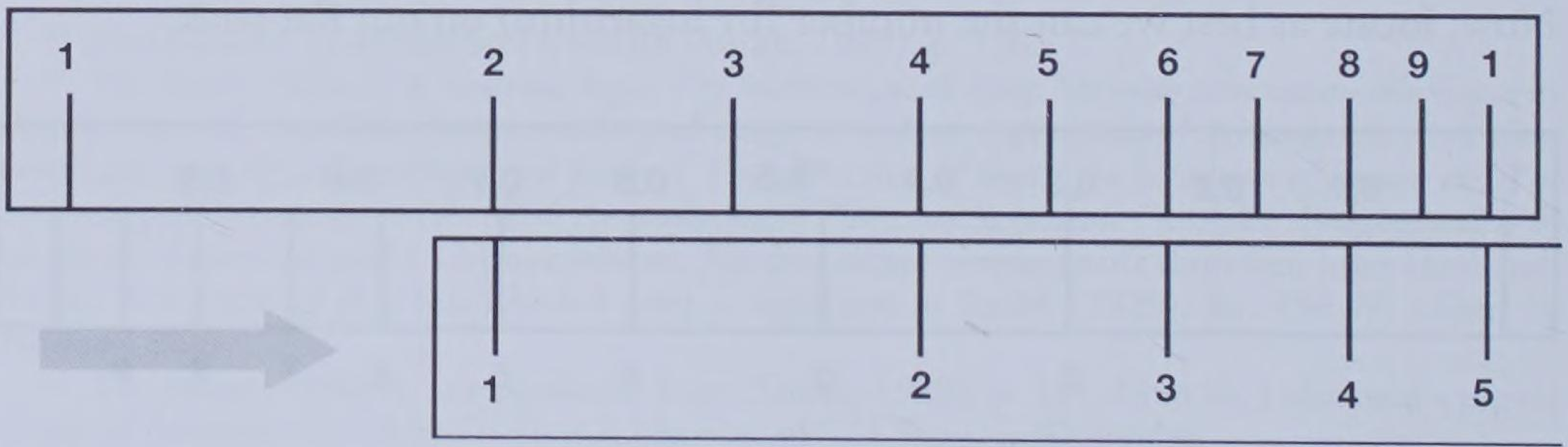

Figure 11.18

Find **3** on the bottom ruler, and its new counterpart on the top ruler is the answer to 2×3. Looking at the top ruler, now, we see that **3** is aligned with **6** (its new counterpart).

Keep in mind that the geometrical progression marked by **2**, **3**, and **4**, for instance, is correlated with their logarithms, 0.30, 0.47, and 0.60 (the more decimal places the better, but here we are working the example with just two). Recall that these logarithms can be reformulated as $10^{0.30}$, $10^{0.47}$, $10^{0.60}$, which equal 2, 3, and 4, respectively. So,

$$2 \times 3 = 10^{0.30102} \times 10^{0.47712}.$$

The exponent Product Rule, recall, tells us:

$$10^{0.30102} \times 10^{0.47712} = 10^{0.30102+0.47712} = 10^{0.77814}$$
$$10^{0.77815} = 6.$$

(0.77814 has been rounded up). In logarithm notation, log 6 = 0.78. Notice that *adding* the logarithms 0.30 and 0.47 is like *multiplying* 2 and 3. What is more, keep in mind that we are just scratching the surface with the use of this machine. NASA engineers did most of their sophisticated calculations for the space program in the 1960s by use of the slide rule!

To divide 9 by 3, say, slide the bottom ruler so that **3** of the bottom ruler is directly underneath **9** of the top ruler:

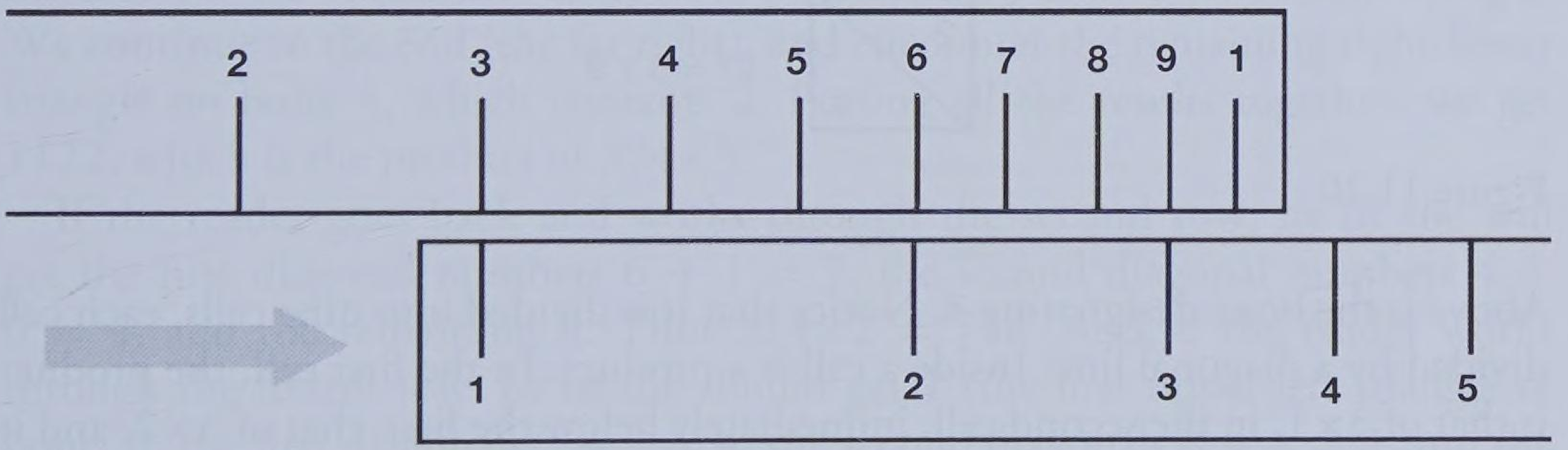

Figure 11.19

Find **1** on the bottom ruler and look to its new counterpart on the top ruler. Here, it is **3**. That is our answer. So, $9 \div 3 = 3$. Notice that *dividing* 9 by 3 is like *subtracting* the logarithms 0.95424 by 0.47712.

Napier invented another calculating machine, which those who used it affectionately called *Napier's Bones*. 'Bones' here referred to the material out of which the device was made—usually bone or ivory. Like the slide rule, the bones (as I will call them) converted multiplication into addition. A single bone (there were several in a set) looked as follows (the bone, left):

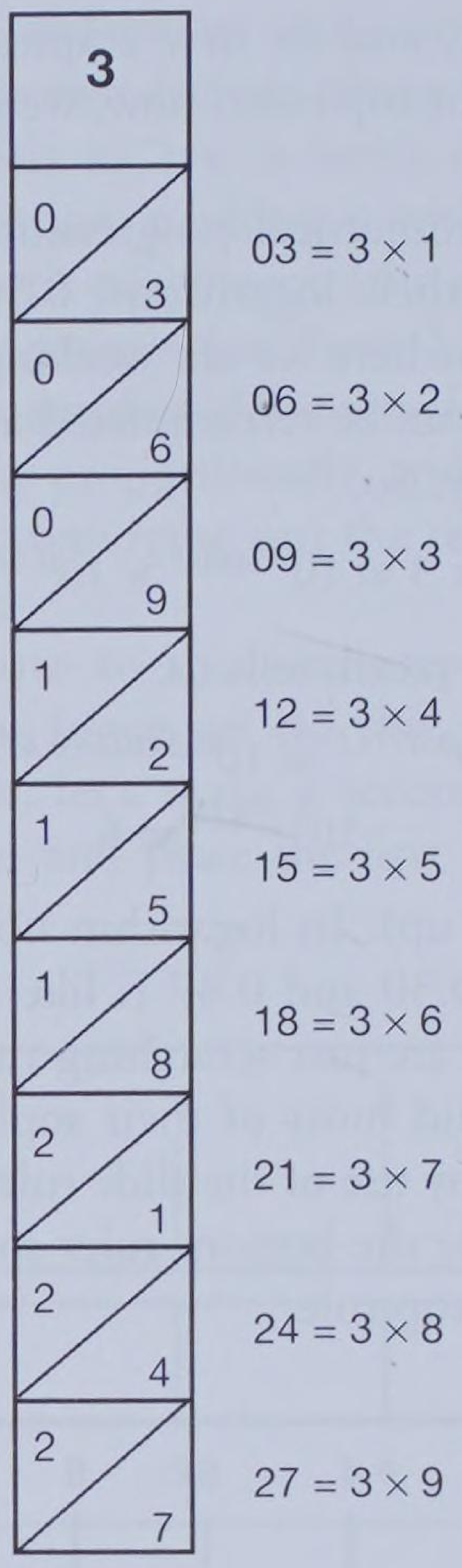

Figure 11.20

Above is the bone designating 3. Notice that it is divided into nine cells, each cell divided by a diagonal line. Inside a cell is a product. In the first cell, the product is that of 3×1, in the second cell, immediately below the first, that of 3×2, and it proceeds all the way to 3×9. When a product is a single-digit number, **0** is placed in the upper-left triangle of the cell. When a product is a double-digit number, as in the cell for the product of 3×5, the first digit is placed in the upper-left triangular division of the cell, the second digit in the lower-right triangle. So, for 15, **1** is placed in the upper-left triangular division, **5** in the lower-right.

A set of bones contained a bone for each integer, 0 to 9, and some contained an additional bone for calculating square roots. Let's say that we needed to multiply 374 by 3. What we do is get the bones for **3**, **7**, and **4**, and put them together, side by side, in the order 374. We read across the third cell of each (since we are multiplying by 3), adding those numbers sharing the same diagonal. This sounds complicated, but it is remarkably simple.

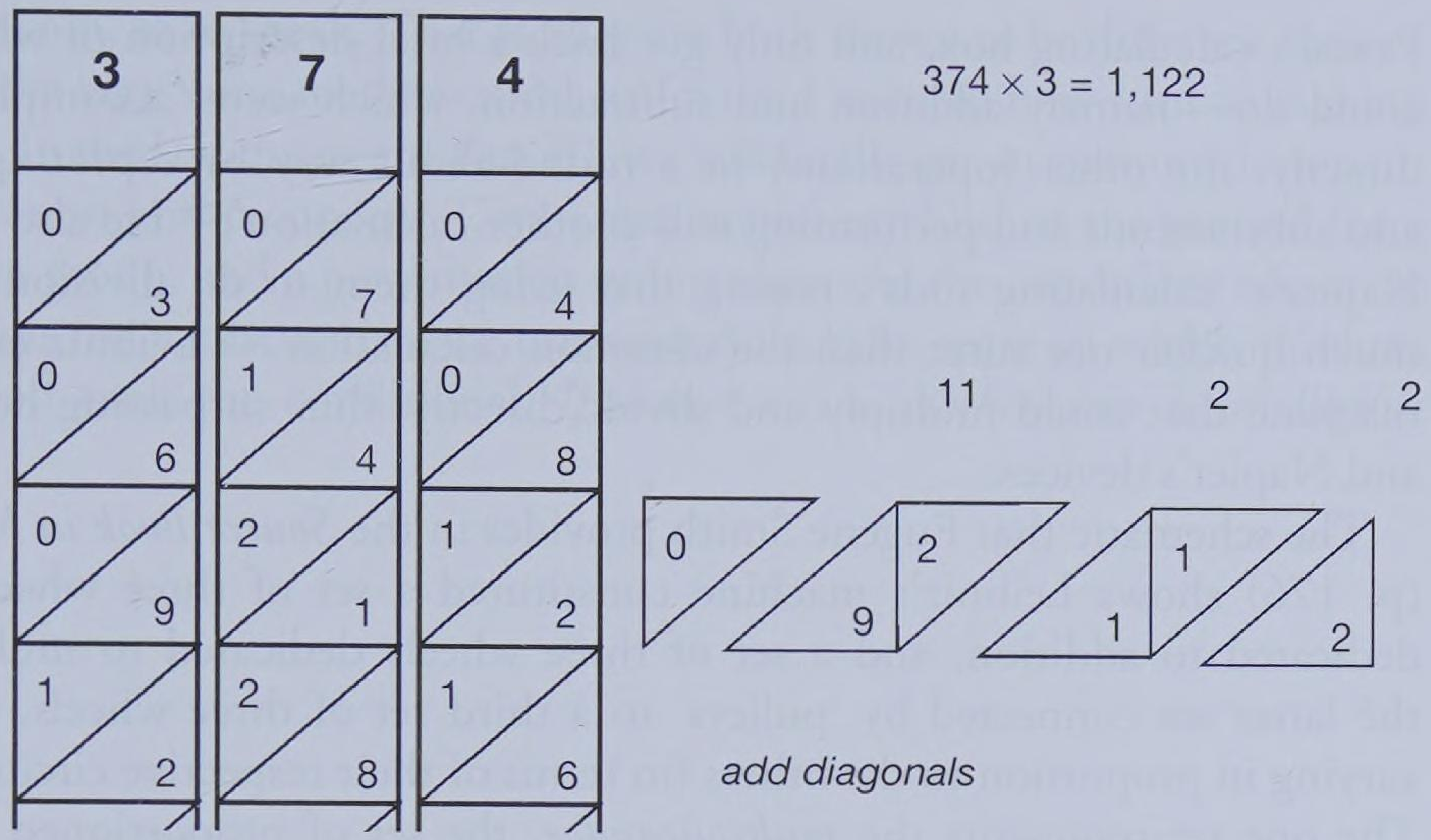

Figure 11.21

Working from left to right, we encounter the first upper-left triangle of the third cell of bone 3, which contains 0. We next encounter the first lower-right triangle of the same cell, which contains 9, and add to it its 'diagonal number', which is the upper-left triangle of the cell immediately adjacent, on bone 7, which contains 2. The sum is **11**. Continuing right, we add the next set of diagonal numbers, 1 and 1, (the first on bone 7, the second on bone 4), their sum being **2**. We continue to the end (the far right), and encounter the remaining right-lower triangle on bone 4, which contains **2**. Putting all the results together, we get **1122**, which is the product of 374×3.

If the reader goes back and works through the second row, he or she will get the first diagonal numbers 6 + 1 = **7**, the second diagonal numbers 4 + 0 = **4**, and the remaining **8**. Thus 374×2 = 748. And, if the reader works through the fourth row, he or she should get **1** (the first upper-left triangle of bone 3), the first diagonals 2 + 2 = **4**, the second diagonals 8 + 1 = **9**, and the remaining lower-right triangle (on bone 4) containing **6**. Thus, 374×4 = 1496. Of course, a bit more needs to done in order to multiply larger numbers, but the procedure turns out to be much easier than multiplying longhand. Both Napier's Bones and the slide rule made calculating easier by translating multiplication into addition.

Both Pascal and Leibniz designed their own mechanical calculating machines. Leibniz said that when he was thinking through the details of his machine, the 'calculating box of Pascal was not known to me'.[23] He wrote to Carcavius about

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[23] Leibniz, 'On His Calculating Machine', translated by Mark Kormes, in Smith (1929), pp. 173–81. The quote is from p. 173. This title is given to it by Smith. The title of the original manuscript was '*Machina artithmetica in qua non additio tantum et subtraction sed et multiplicatio nullo, divisio vero paene nullo animi labore peragantur*' (1685).

Pascal's calculating box, and only got back a brief description of what the box could do—namely addition and subtraction, which were 'accomplished by it directly, the other [operations] in a round-about way by repeating additions and subtractions and performing still another calculation'.[24] He also mentioned Napier's 'calculating rods', noting that using them to do division 'is neither much quicker nor surer than the common calculation'.[25] Leibniz envisioned a machine that could multiply and divide directly, thus surpassing both Pascal's and Napier's devices.

The schematic that Eugene Smith provides in the *Source Book in Mathematics* (p. 176) shows Leibniz's machine constituted a set of three wheels or gears dedicated to addition, and a set of three wheels dedicated to multiplication, the latter set connected by 'pulleys' to a third set of three wheels, each wheel varying in proportion to the others (in terms of their respective circumferences). The one set represents the *multiplicandae*, the set of proportioned wheels the *multiplicantes*. The former is the given number, the latter the number the former is to be multiplied by. So, when Leibniz says, 'Let 365 be multiplied by 124,'[26] 365 is the *multiplicandae* and 124 is the *multiplicantes*.

Recall the slide rule. Proportions were expressed by way of a geometrical progression, determined by a logarithmic function. Leibniz expresses this very idea, but instead of static markings on a flat ruler, the progression is produced by way of rotating wheels of differing circumferences, and the lengths of the pulley chords connecting one wheel to another. So, for example, when one wheel completes a rotation, a second will move some proportion—one-quarter, say. Thus, the ratio or proportion between the first and second wheel is 4 to 1 (the first must completely rotate 4 times in order to completely rotate the second 1 time). Clocks of the period, of course, worked from the same design idea (gears driving minute and hour hands turned at determined ratios). Notice that even here, with respect to time, its measure is conceived in terms of ratios of extended matter (here the gears), which are in turn set into motion.

Whether the gear of a clock, or the turn of one of Leibniz's wheels in his machine, the mathematical conditions are the very same that underwrite the mathematical reality exhibited in the square group, considered above. The calculating machine is possible because the bodies out of which it is made are *inherently* mathematical. The very operations of addition and multiplication, for instance, are expressed in our simply *sliding* two flat sticks of the slide rule together. And, these operations are inherent *in* material bodies. A material world understood thus could not fail to be a mathematical world.

Many of the early combinatorial tables turn out to be tables expressing permutation groups, the latter being genuine mathematical systems. Here emerges an important cross-section of the foundational concepts of arithmetic. What we discover is that this is also where analysis and synthesis are shown to be flip sides of the same coin, so to speak. That is, they are both shown to be different ways of expressing the very same mathematical structure. I want to now turn to making this clearer. In the last chapter of Part III, we will finally get an anatomy lesson on the permutation group concept. Given that matter established the ground for a permutation group, and a permutation group expresses the conditions necessary and sufficient for mathematics, we can understand the sense in which thinkers of the period took the conditional, *If matter exists, mathematics is intelligible* to be true.

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[24] Leibniz (Smith 1929), p. 174.

[25] Ibid., p. 180.

[26] Ibid., p. 176.

12

How Analysis and Synthesis are Related

The group tables that were constructed earlier are instances of what are known as *Latin squares*. A Latin square is an *n* x *n* table, which has *n* elements, where every element appears *n* times, exactly once in each row and column. Recall the earlier table (Table 9.1, see page 144 and below) for mod-3 addition:

Table 12.1

⊕	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

The leftmost column and uppermost row can be taken off (they were used to simply make the table), and the Latin square is revealed:

Table 12.2

0	1	2
1	2	0
2	0	1

This is a 3 x 3 table, with 3 elements (0, 1, 2), in which every element appears three times, found exactly once in each row and column.

Arthur Cayley was clearly aware that the group tables he constructed were Latin squares.[1] Euler also had constructed such tables. In his *De quadratis magicis* (*On Magic Squares*),[2] delivered at St. Petersburg Academy in 1776, Euler works out the mathematical details of how to construct such squares arithmetically. Magic squares are a species of Latin square (recall that Hofmann notes Pascal's interest in them). A magic square is a square in which each element appears only once (this is not the case for Latin squares generally, where given the square is of order *n*, each element must appear *n* times), where the sums of the elements in any row, any column, or even along any diagonal, are identical. One of the earliest (European) images of a magic square is found in Albrecht Dürer's 1514 engraving *Melancholia*:[3]

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[1] A. J. Criley, *Arthur Cayley: Mathematician Laureate of the Victorian Age* (Baltimore: Johns Hopkins University Press, 2006), pp. 186, 416.

[2] Euler, 'De quadratis magicis', first published in *Commentationes Arithmeticae* 2 (1849), pp. 593–602, translated by Jordan Bell.

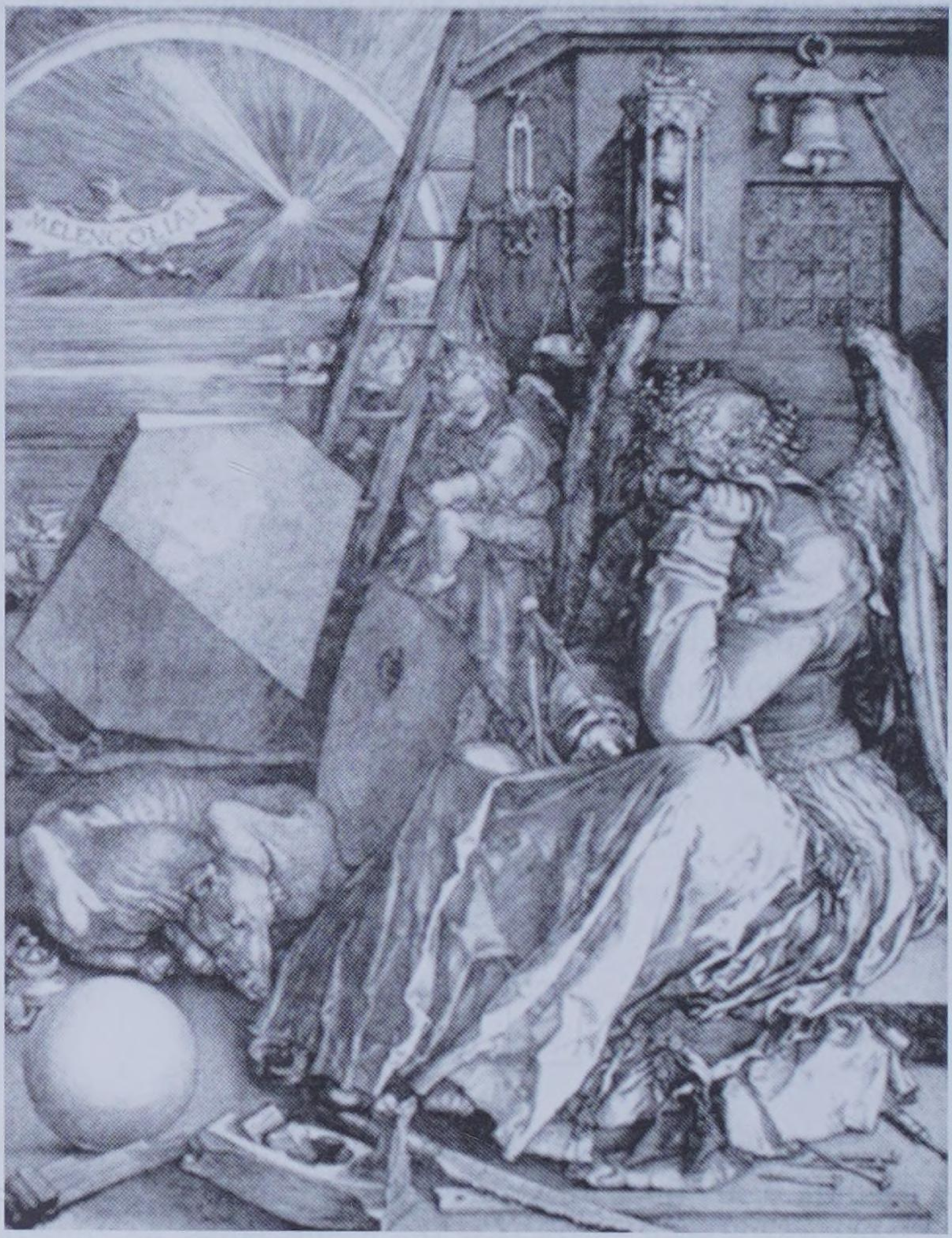

Image 12.1 Albert Dürer: *Melancholia*

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[3] Boyer includes an image of the engraving (p. 325) and a very brief discussion of the magic square. The engraving hangs in the British Museum. Several excellent websites dedicated to the history of mathematics also include the image. It is from one of these sites that I obtained the digital image here, as well as the close-up of the square itself (http://www.agnesscott.edu/LRiddle/women/abstracts/ollerenshaw_Durer.htm).

The magic square is located in the upper right of the engraving, directly above the angel's wing. Here is a close-up of the square:

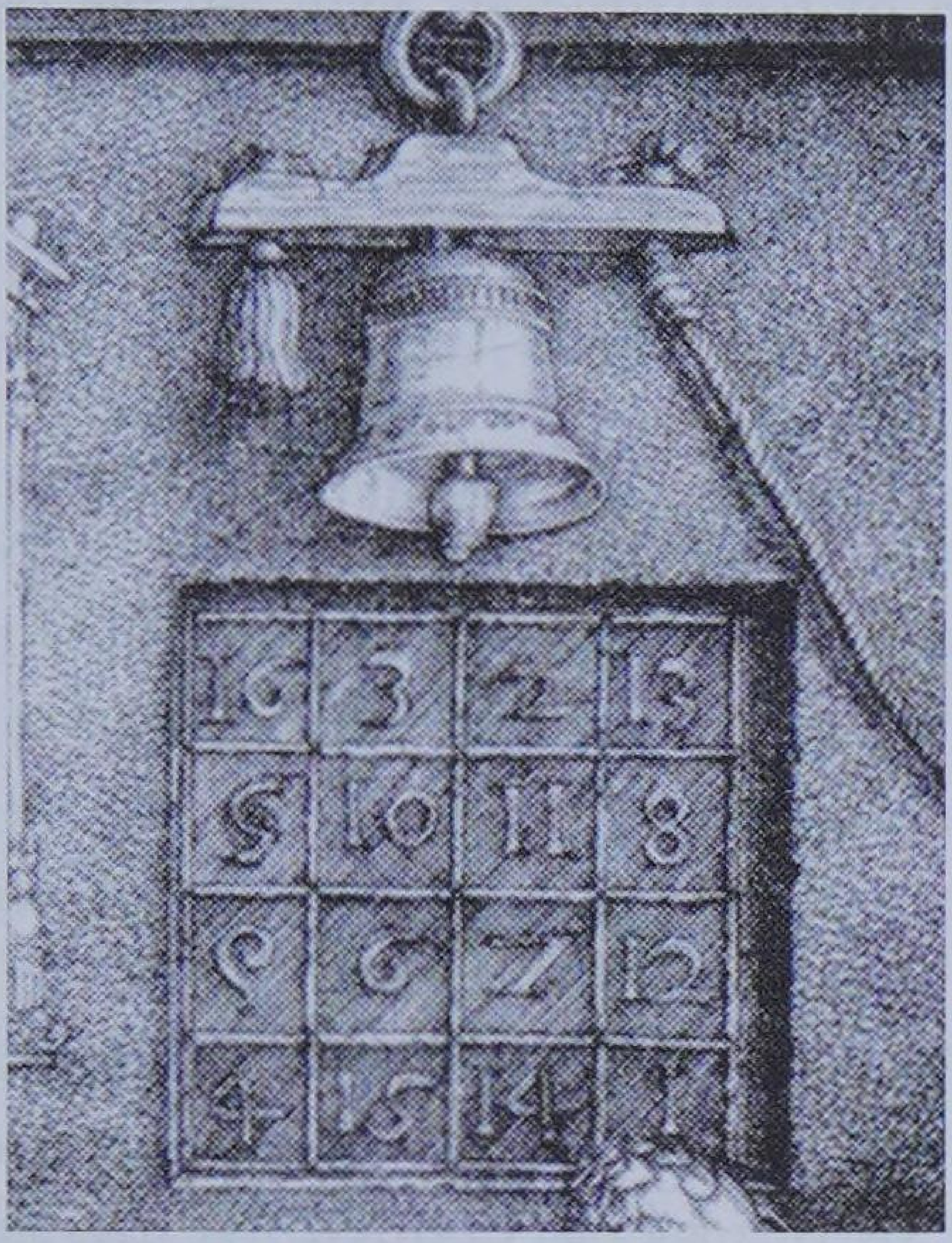

Image 12.2 Close-up of a section of *Melancholia*

The sum of any row, column, or diagonal (and even the sum of the four corners, or the sum of the middle four squares, or the sum of any 2 x 2 group of squares) is 34. Such squares are of interest here for at least two reasons. First, they are related to combinatorial theory as it evolved in the seventeenth century. Second, they embody certain ideas that look to be ancestors of more formal mathematical concepts that become important to my story. For example, with some tables as the number of members increases, so do the sums of the rows and columns, but with other tables (like the magic square), the increase in members is correlated with a limit.

Using the Roman letters *a*, *b*, *c*, *d*, and the Greek letters α, β, γ, δ, Euler constructs a magic square from their combination.

Table 12.3

a α	b β	c γ	d δ
a β	b α	c δ	d γ
a γ	b δ	c α	d β
a δ	b γ	c β	d α

This combinatorial structure would find application in his proof that every integer is a sum of four squares.[4] The *Lemma* is: 'The product of two numbers, each of which is a sum of four squares, may always be expressed as a sum of four squares.'[5] The product of the two numbers (each being a sum of four squares) is:

$$(a^2 + b^2 + c^2 + d^2)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2).$$

Euler then asks us to reconstruct the product as follows, where:

$$\begin{aligned} \mathrm{A} &= \mathrm{a}\alpha + \mathrm{b}\beta + \mathrm{c}\gamma + \mathrm{d}\delta, \\ \mathrm{B} &= \mathrm{a}\beta - \mathrm{b}\alpha - \mathrm{c}\delta - \mathrm{d}\gamma, \\ \mathrm{C} &= \mathrm{a}\gamma + \mathrm{b}\delta + \mathrm{c}\alpha + \mathrm{d}\beta, \\ \mathrm{D} &= a\delta - b\gamma - c\beta - d\alpha. \end{aligned}$$

Notice that the exact array of combinations of Roman and Greek letters in the definitions of A, B, C, and D, is the array depicted in the above magic square (Table 12.3). If we take A, B, C, D as defined, we can now show that:

$$\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2 + \mathrm{D}^2 = (a^2 + b^2 + c^2 + d^2)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2).$$

So, the product of two numbers, each of which is the sum of four squares, is expressible as a sum of four squares.

In *De quadratis magicis*, starting with the number of the square's order—1, 2, 3, 4, . . . up to the *xth* order—Euler says that the sum for each line (row or column) of a magic square will be:[6]

$$\frac{x(1 + xx)}{2}$$

So, if $x = 3$ (that is, the order is 3) for each line the sum is:

$$\frac{3(1 + 9)}{2} = 15.$$

The condition here is that only the integers 1 to 9 are taken to constitute a 3 x 3 square (and each can appear only once in the array). So, what Euler's formula shows is that a proper arrangement of these integers, in respect to their columns and rows, will yield a sum of 15. The following is found in Agrippa's *De occulta philosophia libri tres*, which he took to represent the numerical properties of Saturn:[7]

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[4] Euler, 'Proof that Every Integer is a Sum of Four Squares', translated by E. T. Bell, in *Source Book in Mathematics*, edited by David Eugene Smith (New York: McGraw-Hill, 1929), pp. 91–4.

[5] Ibid., p. 91.

[6] 'De quadratis magicis', The Euler Archive, p. 1.

[7] Such a table can be found in Heinrich Cornelius Agrippa's *De occulta philosophia libri tres* (Paris, 1531). The book was translated into English and printed in 1651. *Three Books of Occult*

Table 12.4

4	9	2
3	5	7
8	1	6

As Euler's formula predicts, each row, column, and diagonal adds up to 15. His formula also shows that for order 4, the sum of each line will be 34 (the *Melancholia* table is an example of this), for order 5, the sum will be 65, and so on. Of course, to make these tables, integers greater than 9 will be allowed.

No doubt Pascal's interest in Latin and magic squares found its way into his *Triangle Arithmétique*, published in 1665. The history of the 'triangle' (today called 'Pascal's triangle') as an important table of combinatorial numbers has been traced back far beyond the early modern period—versions of its use as a combinatorial table found as early as 300 BC in the *Bhagabati Sutra* of the Jains.[8] In Europe, we find serious treatments of the properties of the table as early as Tartaglia's *General Trattato* (1556), Cardano's *Opus Novum* (1570), and Mersenne's *Harmonicorum Libri XII* (1636).[9] A. W. F. Edwards notes an interesting connection between Mersenne's book and Pascal's.

When Pascal was working on this material, Edwards writes:

> We know that the Pascals, father and son, visited him [Mersenne], so that it comes as no surprise to find that Pascal's format for the Arithmetical Triangle is the same as Mersenne's; we know that he corresponded with the Lullist Athanasius Kircherus, whose *Musurgia universalis* of 1650 repeats many of Mersenne's results, even to the extent of using the same numerical examples. Leibniz owned that Lullism had had some influence on his *Dissertatio de Arte Combinatoria* of 1666.[10]

The triangle can be easily constructed. Starting with a basic triangle of ones:

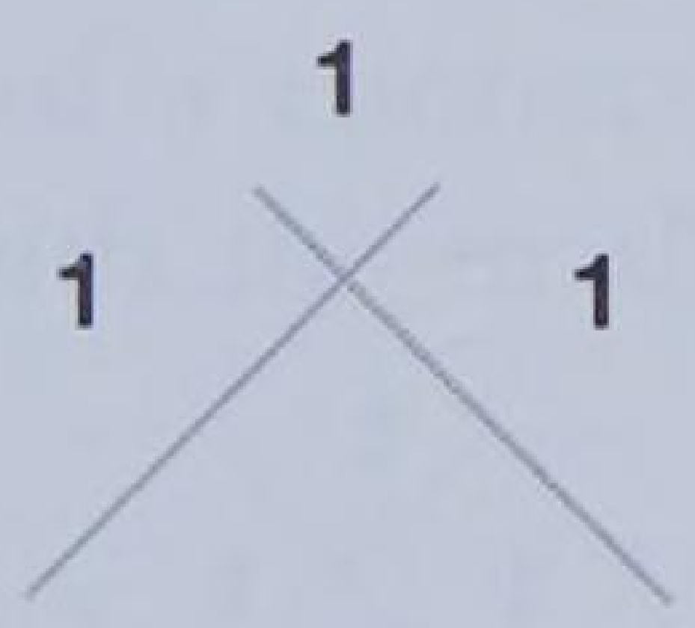


Figure 12.1

Philosophy, translated by James Freake (London: R. W. for Gregory Moule, 1651). The table is found in Book II, Chapter xvii, 'Of the Table of the Planets, their Vertues, forms, and what Divine names, Intelligences, and Spirits are set over them'. An online version of the text could be found at the time of writing at http://en.wikisource.org.

[8] A. W. F. Edwards, *Pascal's Arithmetical Triangle* (Baltimore: Johns Hopkins University Press, 1987), p, 27.

[9] Ibid., pp. 39, 44, 46.

[10] A. W. F. Edwards, *Pascal's Arithmetical Triangle*, p. 47.

Building the next row of numbers is easy. Always beginning and ending with 1 (for reasons to be given shortly), we simply add the numbers of the row immediately above the one we are constructing. So, building now the next row, starting with 1, the next number in the sequence (to its right) will be determined by adding the 1's immediately above. Thus:

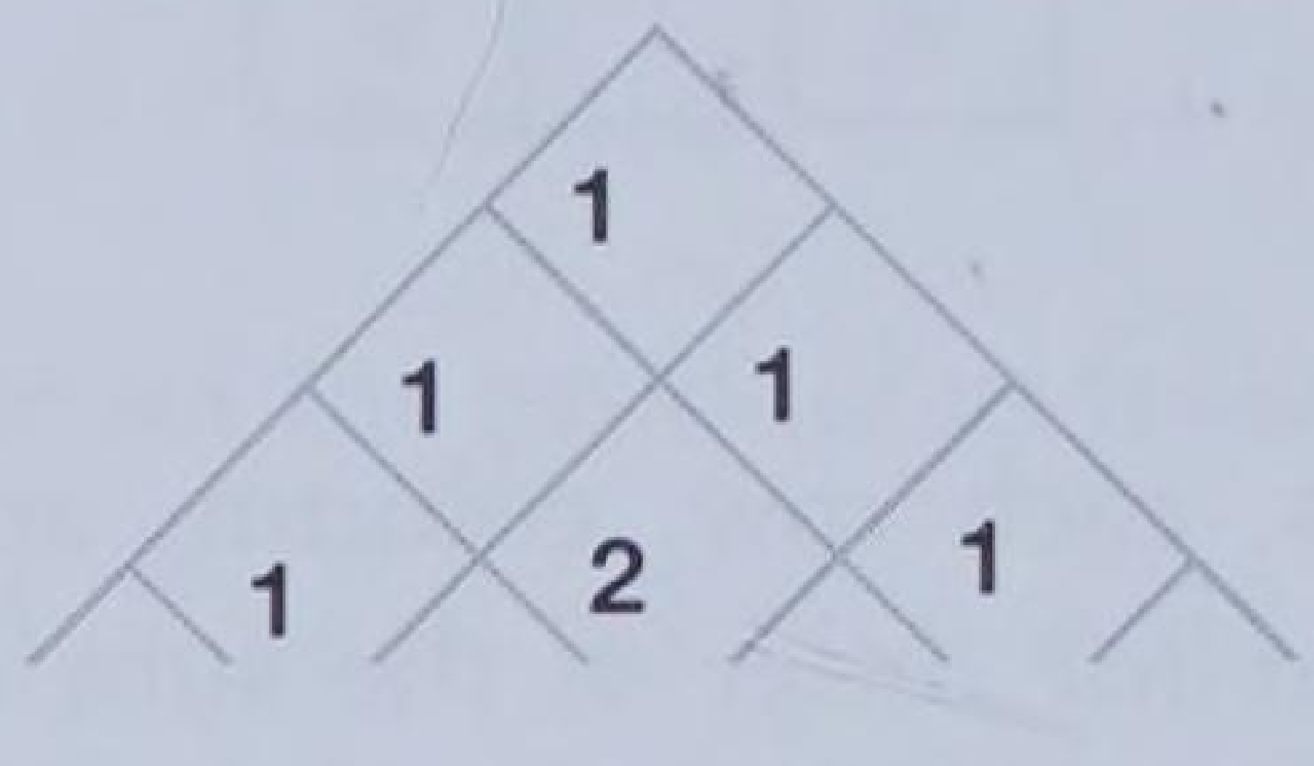


Figure 12.2

Building the next row:

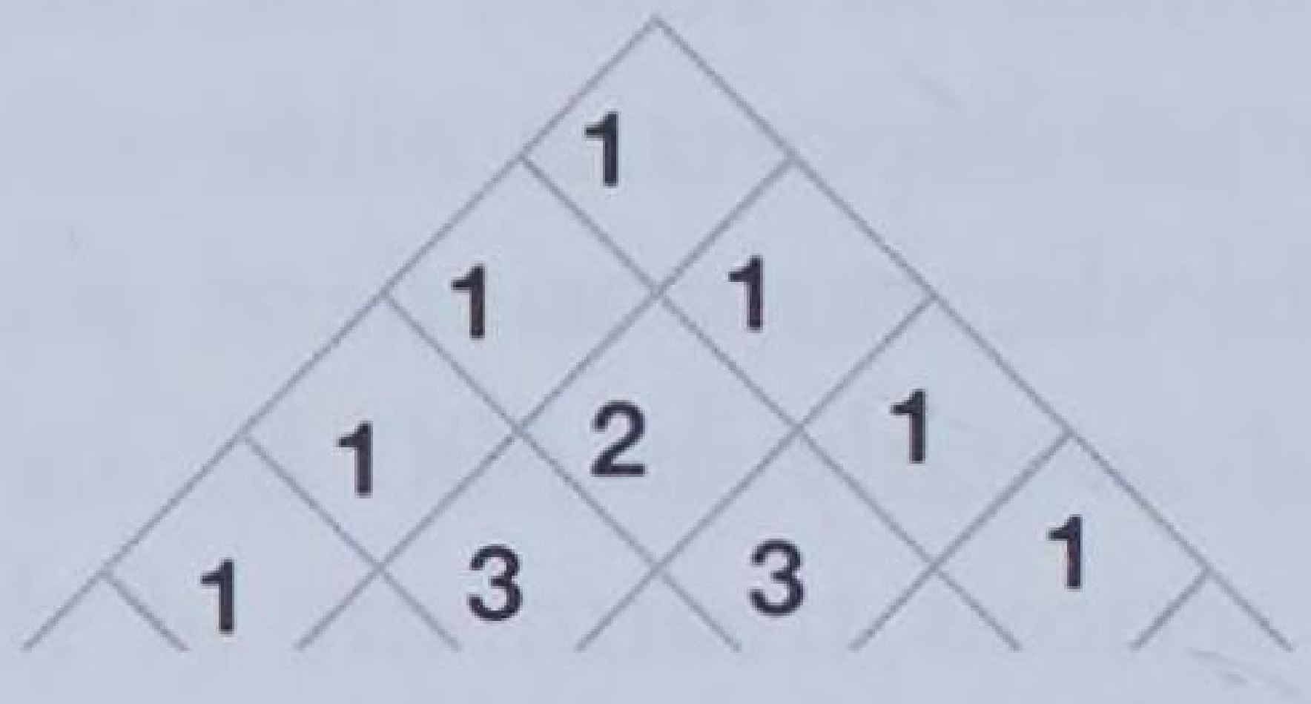


Figure 12.3

And, the next:

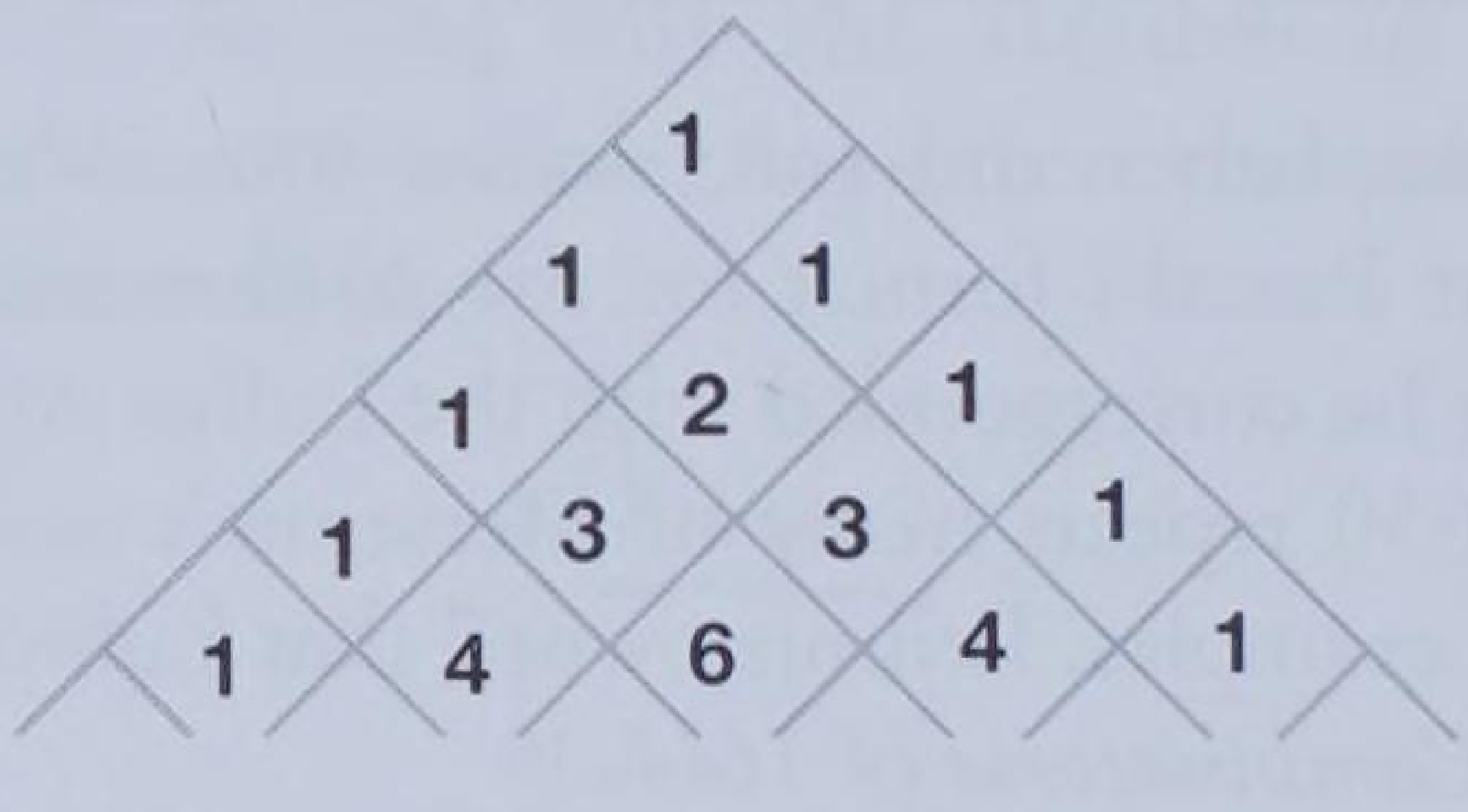


Figure 12.4

And, the next:

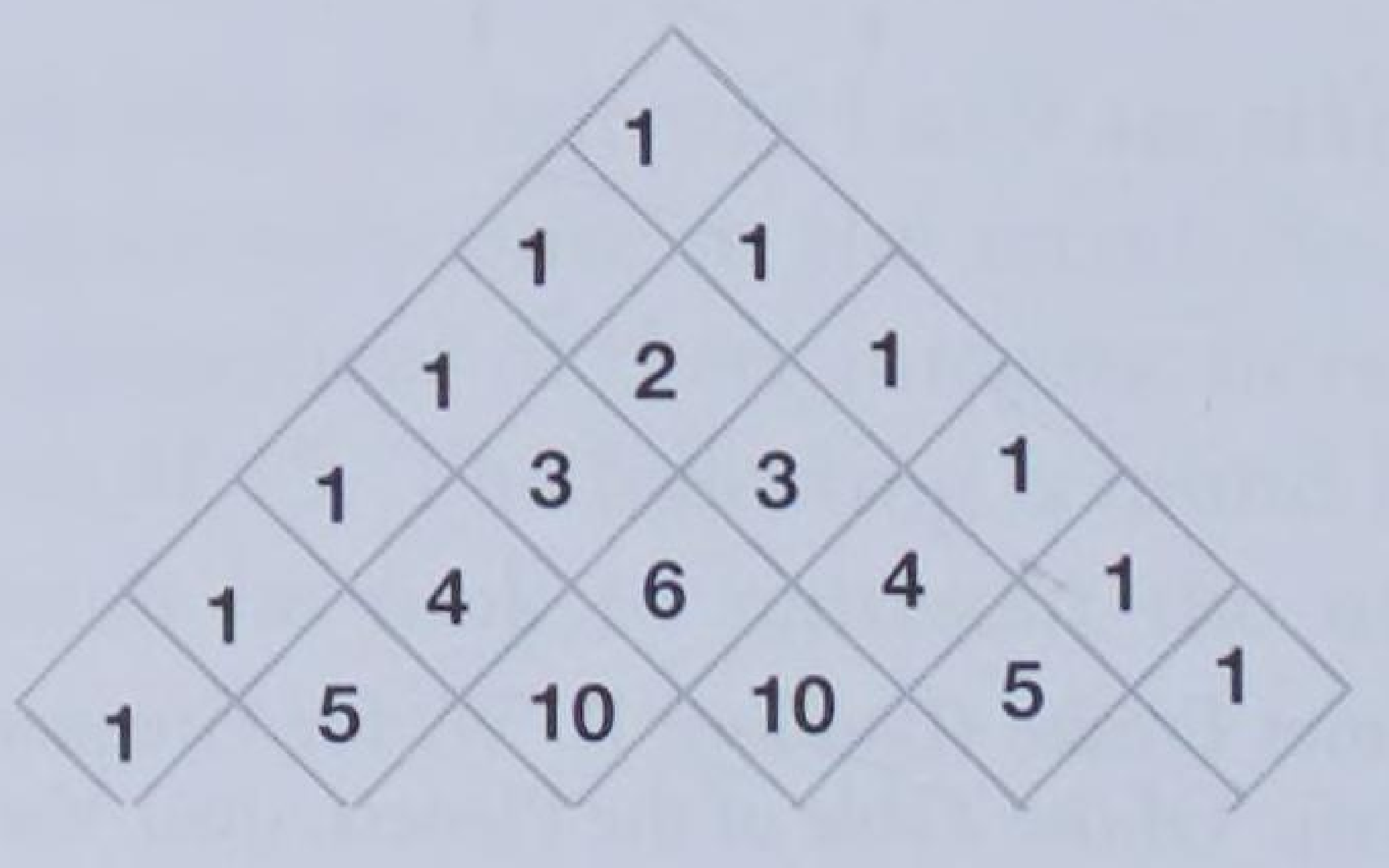


Figure 12.5

There is much of interest in this table (which can be extended as far as we would like). For a start, as was just mentioned, the horizontal rows can be used as 'combinatorial' numbers. Here is an example: suppose that we have three books, and suppose that we wish to take the books in different combinations—one at a time, two at a time, and so on. Here is what we find:

> There is **1** way to take 0 books at a time,
> There are **3** ways to take 1 book at a time,
> There are **3** ways to take 2 books at a time,
> There is **1** way to take 3 books at a time.

This is easy to understand. Let the three books be A, B, and C.

> There is **1** way to take 0 books at a time:
> take nothing.
> There are **3** ways to take 1 book at a time:
> take A, or take B, or take C.
> There are **3** ways to take 2 books at a time:
> take AB, or take BC, or take AC.
> There is **1** way to take 3 books at a time:
> take ABC.

Notice that the 'ways' to take the books are **1**, **3**, **3**, **1**, which is a row of the triangular table. Given four books, we find:

> There is **1** way to take 0 books at a time,
> There are **4** ways to take 1 book at a time,
> There are **6** ways to take 2 books at a time,
> There are **4** ways to take 3 books at a time,
> There is **1** way to take 4 books at a time.

Here, the combinatorial numbers are **1**, **4**, **6**, **4**, **1**, which again is a row of the triangular table. The reader can now easily determine what the combinations will be given five books, six books, and so on.

Pascal also works out details of a connection between the triangular table and binomial expansion. Start with the mathematical formula $(x + 1)^2$. The expansion proceeds as follows:

$$\begin{aligned}(x+1)(x+1) &= x^2 + x + x + 1\\ &= x^2 + 2x + 1.\end{aligned}$$

The coefficients are **1**, **2**, **1**, which is a row of the triangular table (as is customary, when 1 is a variable's coefficient, it is usually not written). Now consider the formula $(x + 1)^3$:

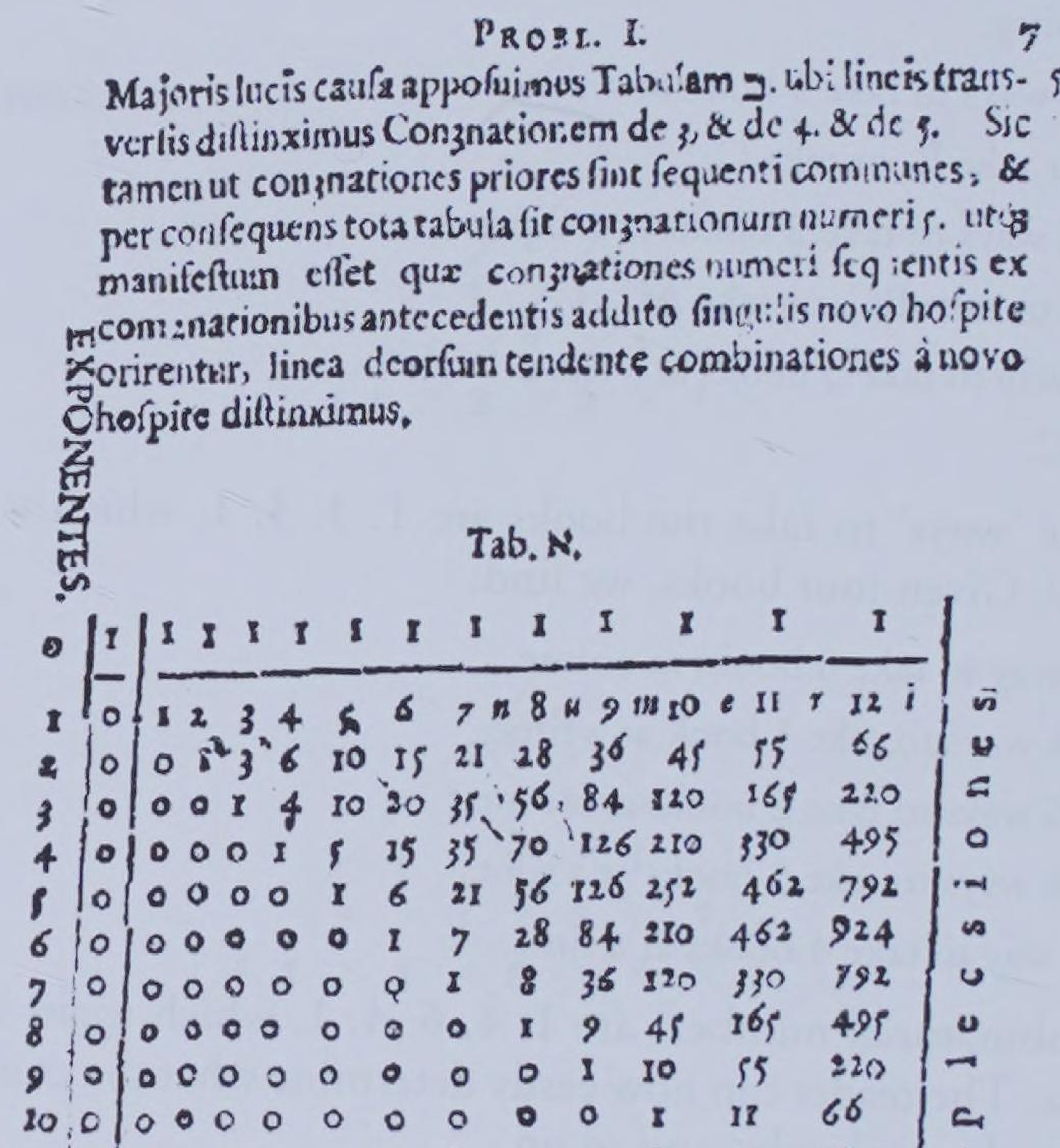
$$\begin{aligned}(x+1)(x+1)(x+1) &= (x+1)(x^2 + 2x + 1)\\ &= x^3 + 3x^2 + 3x + 1.\end{aligned}$$

The coefficients are **1**, **3**, **3**, **1**, which again is a row of the table. Now consider $(x+1)^4$:

$$(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1.$$

The combinatorial numbers are **1**, **4**, **6**, **4**, **1**, which is a row of the table. The reader can now easily determine the expansion of $(x+1)^5$, $(x+1)^6$, and so on.

Apparently unaware of Pascal's book, Leibniz makes some of the very same connections in his *Dissertatio de Arte Combinatoria*,[11] published in 1666, just a year after Pascal's *Triangle Arithmétique*. We find the triangle, table ℵ, below.



PROBL. I. 7

Majoris lucis cauſa appoſuimus Tabulam ב. ubi lineis transverſis diſtinximus Con3nationem de 3. & de 4. & de 5. Sic tamen ut con3nationes priores ſint ſequenti communes; & per conſequens tota tabula ſit con3nationum numeri 5. utque manifeſtum eſſet quæ con3nationes numeri ſequentis ex com3nationibus antecedentis addito ſingulis novo hoſpite orirentur, linea deorſum tendente combinationes à novo hoſpite diſtinximus. 5.

EXPONENTES.

Tab. ℵ.

0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6	7 n	8 u	9 m	10 e	11 r	12 i
2	0	0	1	3	6	10	15	21	28	36	45	55	66
3	0	0	0	1	4	10	20	35	56	84	120	165	220
4	0	0	0	0	1	5	15	35	70	126	210	330	495
5	0	0	0	0	0	1	6	21	56	126	252	462	792
6	0	0	0	0	0	0	1	7	28	84	210	462	924
7	0	0	0	0	0	0	0	1	8	36	120	330	792
8	0	0	0	0	0	0	0	0	1	9	45	165	495
9	0	0	0	0	0	0	0	0	0	1	10	55	220
10	0	0	0	0	0	0	0	0	0	0	1	11	66
11	0	0	0	0	0	0	0	0	0	0	0	1	12
12	0	0	0	0	0	0	0	0	0	0	0	0	1
*	0	1.	3.	7.	15.	31.	63.	127.	255.	511.	1023.	2047.	4095.
+	1.	2.	4.	8.	16.	32.	64.	128.	256.	512.	1024.	2048.	4096.

Complexiones

Complexiones ſimpliciter* (ſeu ſumma Complexionum dato exponente) addita unitate, quæ coincidunt cum terminis progreſſionis geometricæ duplæ.†

Tab. ב.

Image 12.3 Leibniz, Table ℵ, Dissertatio de Arte Combinatoria

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[11] The image comes from the Gallica website.

The exponents (*exponentes*) flank the left side of the table. The combinatorial numbers (here the table is on its side) descend down each column, the 'apex' of the triangle located at the upper-left corner of the table. Leibniz calls these numbers complexions (*complecsiones*). In addition to the sorts of connections made by Pascal, there is an interesting connection that Leibniz makes worth mentioning. In the row and corresponding footnote, designated by †, he provides the sums of each column, noting that they are equivalent to their corresponding exponent of base of 2 (*duplae*). The insight is (I extend the example to only 2^4):

Table 12.5

Exponent	Column	Sum (of column)
2^0	1	1
2^1	1+1	2
2^2	1+2+1	4
2^3	1+3+3+1	8
2^4	1+4+6+4+1	16

In a mathematical fragment *De Dyadicis*,[12] written roughly fourteen years after the *Dissertatio*, Leibniz records the following connection between base 2 numbers and binary notation. He writes:

1	1 =	2^0
10	2	2^1
100	4	2^2
1000	8	2^3
10000	16	2^4
100000	32	2^5
1000000	64	2^6

If we go back to table ℵ of the *Dissertatio* (Image 12.3) we will see where the *De Dyadicis* connection very likely has its origin. Consider the relevant portion of the table up close:

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[12] Leibniz, 'De Dyadica', *De Mathematical Schriften von Gottfried Wilhelm Leibniz*, vol. 3, von C. I. Gerhardt (New York: George Olms Verlag, 1978), p. 574. I am indebted to Greg Brown for helping me locate this document.

0	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6
2	0	0	1	3	6	10	15
3	0	0	0	1	4	10	20
4	0	0	0	0	1	5	15
5	0	0	0	0	0	1	6

Figure 12.6

Where the exponents are listed to the far left, the darkened zeros and terminal one (also underlined) to their immediate right correspond to the *De Dyadicis* table (though they end instead of begin with 1). Although I will not go into it further, it is worth mentioning that the use of V and Λ, which functioned as identity elements in the mathematical structure (Boolean algebra) Leibniz developed, were taken as the universal class (the class of everything) and the null-class (the class of nothing). These, in his attempts at working out a binary system, correspond to 1 and 0 respectively. No doubt, these would have their metaphysical counterparts in the Parmenidean dichotomy of the categories of *Being* and *Non-Being*.[13] I mention this because it is another example of Leibniz's using mathematical concepts to illuminate his philosophical view.

Through Oldenburg, Newton communicated with Leibniz about binomial expansion.[14] In a 24 October 1676 letter, Newton sees the pyramid's combinatorial numbers connected to exponents, not of base 2 as did Leibniz, but to exponents of base 11. Newton writes that 'these were the figures of the powers of the eleventh number, namely 11^0, 11^1, 11^2, 11^3, 11^4. That is first 1, then 1, 1, thirdly 1, 2, 1, fourth, 1, 3, 3, 1, fifth 1, 4, 6, 4, 1 &c.'[15] What Newton means here is:

$$11^0 = 1$$

$$11^1 = 11$$

---

[13] This might be seen as a challenge to the claim that matter is required for instituting mathematics. But, as I understand Leibniz, 1 could easily be taken to represent the active principle and 0 the passive principle, where the latter is matter (that is, what I have been calling matter, which is Cartesian extension).

[14] Isaac Newton, 'On the Binomial Theorem for Fractional and Negative Exponents', translated by Eva Sanford, in *Source Book in Mathematics*, edited by David Eugene Smith (New York: McGraw-Hill, 1929), pp. 224–8. The letters are from 13 June 1676 and 24 October 1676.

[15] Newton, (Smith 1929), p. 226.

$$\begin{aligned} 11^2 &= 121 \\ 11^3 &= 1,331 \\ 11^4 &= 14,641, \end{aligned}$$

and so on. The number 14,641 is the fifth row of the pyramid: 1, 4, 6, 4, 1. So, where Leibniz had connected the *sum* of the numbers constituting a row with 2 raised to some power, Newton connected the row itself, taken *as a number*, with 11 raised to some power. Both connections led to general mathematical formulations that could generate the combinatorial numbers.[16]

John Wallis (1616–1703), who translated into English (from Latin) Newton's work on the binomial theorem, including it in his *De algebra tractatus historicus & practicus*, lauded over Newton's discovery that if the exponents were fractions—for example, $(x + 1)^{1/2}$—the expansion could yield an infinite series, which, Wallis notes, can 'be continued as far as we please', and where 'the farther it is continued, the more exactly doth it represent the quantity sought'.[17] Wallis says that both he and Newton agreed that such a series ought to be 'called by the name of *Infinite Series*, or *Converging Series*'.[18] He goes on to say, 'Thereby imitating, the designation of some particular quantity, by a regular Progression or rank of quantities, continually approaching to it; and which, if infinitely continued, must be equal to it.'[19] I will explain.

The idea here is not very difficult. If we began, for instance, with 1/1, and added to that 1/2, and then to that, 1/3, where the series looks thus:

$$1/1 + 1/2 + 1/3 + 1/4 + 1/5 + \ldots$$

the series would slowly approach 2, or in Wallis and Newton's terms, we should say that the series *converges* on 2. The more we continued the series, the closer to 2 we would get. An expansion of an equation with fractional exponents would yield terms like the above series that would converge on a number. The idea here is importantly connected to the concept of a *limit*, which roughly speaking tells us that as an x value approaches some number n, say, the y value (determined by some function $f(x)$, where $f(x) = y$) approaches some number m. Ancestors of the limit concept can be found much earlier, for example, in Napier's work on logarithms.

In Part II, I mentioned Leibniz's use of his combinatorial theory to determine the number of Aristotelian syllogistic forms—determined by the combination of categorical propositions, and then limited by the concepts of mood and figure. Leibniz, in fact, considers several practical uses of the combinatorial theoretic machinery. For example, on page 47 of the *Dissertatio*, he notes how colours could be produced (say, by a paint company) from the various combinations (representing proportions) of the primary colours. On page 66, he shows how various combinations of 6 '*voces*' (which he lists as *Ut*, *Re*, *Mi*, *Fa*, *Sol*, *La*) could produce a variety of melodies, where the arrangements of notes, as in the case for colous, are determined by algorithm. In one of his brief discussions of Lull (there are several), he considers the idea of a universal language machine (*scriptura universalis*),[20] which leads to his considering a machine based on combinations, which through an algorithm is able to reproduce verses, for instance, from the works of Augustus, Constantine, and Ovid.[21]

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[16] In correspondence, Leibniz and Jean and Jacques (I) Bernoulli (the letter dates are 16 May 1695 and 8 June 1695) work up a general formula for polynomial (in contrast to simply binomial) expansion. (Smith 1929), pp. 229–31.

[17] John Wallis, 'Wallis and Newton', edited by David Eugene Smith, in (Smith 1929), p. 222.

[18] Ibid., p. 219.

[19] Ibid.

Now, what is it about the combinatorial mathematics that has caught the attention of Pascal, Leibniz, the Bernoullis, Newton, and Wallis, to name only a few—aside from the fact that it can help make someone a better gambler, or mixer of paint?[22] What is the *mathematical* insight? To offer a possible answer, consider the following.

Suppose that there are three objects. The first is ***red***, the second ***white***, and the third ***blue***. Let the objects be ordered (relative to one another) thus:

red white blue

From this arrangement, let's rearrange them. So:

white blue red

Moving from the first arrangement to the second is a *transformation*. In this case ***red*** is replaced with ***white***, ***white*** with ***blue***, and ***blue*** with ***red***.[23] Keeping the transformation as a unit, we will put the arrangements together as follows:

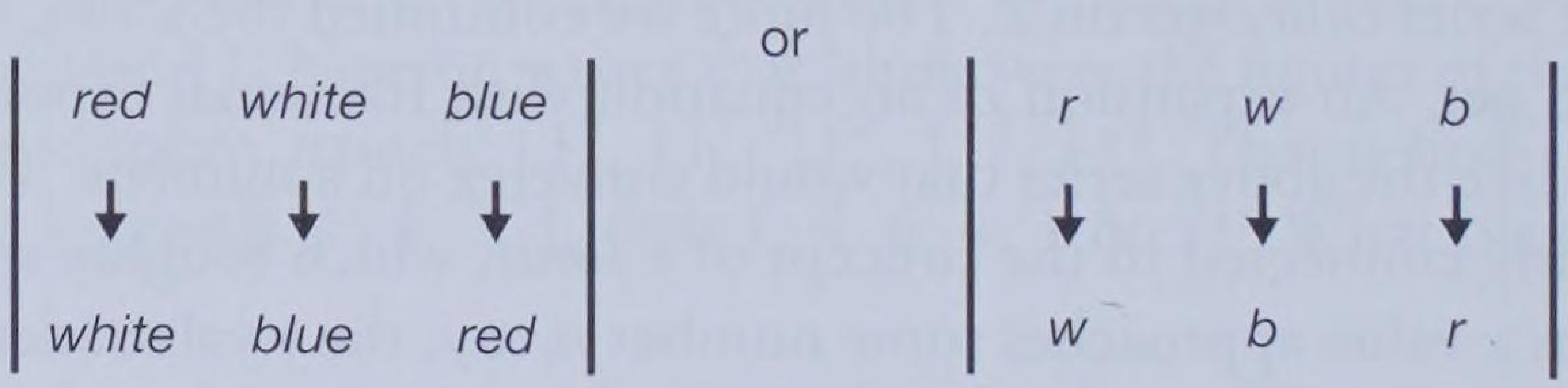

Figure 12.7

Let this transformation be called A. (I will drop the arrows.)

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[20] Dissertatio, p. 43.

[21] He does this on pages 66, 72, and 74 respectively.

[22] It is well known that Pascal, for instance, was prompted to put together the *Arithematique Triangle* from a request to show how one might better understand the probabilities of playing a game of dice.

[23] I am following Cayley, though he calls the replacement procedure *substitution.* Arthur Cayley, 'Desiderata and Suggestions: No. 1 The Theory of Groups,' *American Journal of Mathematics*, vol. I, no. 2 (1878), pp. 50–2.

So,

$$\mathbf{A} = \left|\begin{array}{ccc} r & w & b \\ \\ w & b & r \end{array}\right|$$

Figure 12.8

Now, let the next transformation be:

$$\mathbf{B} = \left|\begin{array}{ccc} r & w & b \\ \\ r & b & w \end{array}\right|$$

Figure 12.9

Here, the transformation goes from (top to bottom) ***red*** to ***red***, ***white*** to ***blue***, and ***blue*** to ***white***. That ***red*** remains unchanged does not count against **B**'s being a genuine transformation. Now, consider the rest:

$$\mathbf{C} = \left|\begin{array}{ccc} r & w & b \\ \\ b & r & w \end{array}\right| \qquad \mathbf{D} = \left|\begin{array}{ccc} r & w & b \\ \\ b & w & r \end{array}\right|$$

$$\mathbf{E} = \left|\begin{array}{ccc} r & w & b \\ \\ r & w & b \end{array}\right| \qquad \mathbf{F} = \left|\begin{array}{ccc} r & w & b \\ \\ w & r & b \end{array}\right|$$

Figure 12.10

There are no other possible transformations.

Let ⊕ be an operation that combines the effects of the transformations. So, if we want to figure out **A** ⊕ **B**, we take the result of the **A** transformation and apply **B** to it. What we should have is a combination of the **A** and **B** transformations. Thus:

$$\mathbf{A} \oplus \mathbf{B} = \left|\begin{array}{ccc} r & w & b \\ \\ b & w & r \end{array}\right|$$

Figure 12.11

We arrived at this as follows:

$$A = \left|\begin{array}{c} r\ w\ b \\ w\ b\ r \end{array}\right| \text{ and } B = \left|\begin{array}{c} r\ w\ b \\ r\ b\ w \end{array}\right|.$$

A transforms ***red*** to ***white***, ***white*** to ***blue***, and ***blue*** to ***red***. So, if we start with the order

r w b,

A transforms this to

w b r.

B transforms ***red*** to ***red***, ***white*** to ***blue***, and ***blue*** to ***white***. So, in applying **B** to the result of **A**—that is, to **w b r**—the result is transformed to

b w r.

Looking at our transformations, A ⊕ B is identical to D:

$$\mathbf{D} = \left|\begin{array}{ccc} r & w & b \\ b & w & r \end{array}\right|$$

Figure 12.12

So, A ⊕ B = D. We can go on to combine all the transformations, {A, B, C, D, E, F}, using ⊕. Here is the table:

Table 12.6

⊕	E	F	D	A	C	B
E	E	F	D	A	C	B
F	F	E	A	D	B	C
D	D	C	E	B	F	A
A	A	B	F	C	E	D
C	C	D	B	E	A	F
B	B	A	C	F	D	E

1) The set S of transformations is closed with respect to $\oplus$.
2) The operation $\oplus$ is associative (I will not show that here).
3) S has an identity element, namely, **E**. (e.g., $\mathbf{A} \oplus \mathbf{E} = \mathbf{A}$).
4) Each member of the set has an inverse. (e.g., since $\mathbf{A} \oplus \mathbf{C} = \mathbf{E}$, **C** is the inverse of **A**, and vice versa).

Since the above meets the four conditions of a group, the set S, the operation $\oplus$, and the identity element **E** form a group (S, $\oplus$, **E**). This is called a *permutation group*. Cayley proved that *every finite group is isomorphic to some permutation group.*[24]

The 'additivity' is brought out in our combining the elements of the system. In the mod-3 group, the elements are 0, 1, 2; in the square group, the elements are the moves **Q**, **H**, **T**, **E**; and in the tricolour group, the elements are the transformations **A**, **B**, **C**, **D**, **E**, **F**. If combination were impossible, it seems that a mathematical reality would also be impossible. Earlier, I argued that combinability requires (presupposes) some notion of divisibility. There must be some sense in which we can say that there are at least *two* things in play, and that these things can be combined, even if combination never actually occurs.

Wittgenstein says in the *Tractatus*, when talking about the (pseudo-)*concept* of object, that 'Just as we cannot think of spatial objects at all apart from space, or temporal objects apart from time, so we cannot think of *any* object apart from the possibility of its connexion with other things.'[25] I take it that objects, understood in light of this logical concept, need not actually be combined in this or that way. They simply must be *combinable*, or be *capable* of entering into relationships with other objects. The combinability feature, we might call it, is inherent in the very concept of *object*, and, subsequently, in the concept of *material object*. Wittgenstein goes on to say, for instance, that 'Objects contain the possibility of all states of affairs,'[26] and that 'The possibility of its occurrence in atomic facts is the form of the object.'[27] We could not conceive of a (finite) body that could not in principle be combined with or related to another, or we could not conceive of the parts of a body, which we made by division, that could not be related or combined with one another. In Cartesian terms, as we noted elsewhere, we might say that bodies are not *really distinct* from one another (nor are the parts of bodies really distinct from other parts), since we cannot conceive of them as existing independently of *all* others, which would align with Descartes's remark to Gassendi about geometry's dealing with *modes* of extension (and not with extension *sans* modal division).[28]

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[24] Arthur Cayley (1878), pp. 50–52. According to Sujoy Chakraborty and Munibur Rahman Chowdhury, Caylely seems poised to make the proof much earlier, in a paper published in 1854. 'Arthur Cayley and the Abstract Group Concept', *Mathematics Magazine*, vol. 78, no. 4, October 2005, p. 273.

[25] Ludwig Wittgenstein, *Tractatus Logico-Philosophicus*, translated by C. K. Ogden (New York: Routledge, 1922), proposition 2.0121, p. 33.

[26] Ibid., proposition 2.014, p. 35.

[27] Ibid., proposition 2.0141, p. 35.

To say that we could conceive of a finite body that is absolutely *un*relatable to any other would be like saying that we can conceive of a square as existing, in some absolute sense, independently of all other Euclidean figures. It seems to me that if we have a square, this 'brings with it' (whether we are aware of it or not) everything else in Euclidean geometry, including all the conditions necessary and sufficient for every other possible figure.[29] Although (modally) distinct, there is a sense in which triangles are inherent in squares, circles in triangles, squares in triangles, triangles in circles, and so on—this is so for the simple reason that the very conditions that make up any one of them make up the others. Likewise, we cannot *conceive* of a finite body existing, in some absolute sense, independently of all others (given that one of the things that determines a finite body is its shape). Given that a material object is (trivially) an object, as Wittgenstein puts it in the passage cited above, we cannot think of it apart from the possibility of its connection with other things.

About his own conception of matter, Descartes says in Chapter 6 of *The World*:

> On the other hand, let us not also think that this matter is the 'prime matter' of the philosophers, which they have stripped so thoroughly of all its forms and qualities that nothing remains in it which can be clearly understood. Let us rather conceive it as a real, perfectly solid body which uniformly fills the entire length, breadth, and depth of this huge space in the midst of which we have brought our mind to rest. Thus each of its parts always occupies a part of that space which it fits so exactly that it could neither fill a larger one nor squeeze into a smaller; nor could it, while remaining there, allow another body to find a place there.
>
> Let us add that this matter may be divided into as many parts having as many shapes as we can imagine, and that each of its parts is capable of taking on as many motions as we can conceive. Let us suppose, moreover, that God really divides it into many such parts, some larger some smaller, some of one shape and some of another, however we care to imagine them.[30]

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[28] For more on divisibility and our not being able to conceive one region of extension apart from all other regions, see Tad Schmaltz, 'Spinoza on the Vacuum', *Archiv für Geschichte der Philosophie* 81:2 (1999), pp. 174–205. Alan Nelson and I have coauthored a piece that deals with the apparent contradiction that Descartes involves himself in, in requiring the divisibility *and* indivisibility of extension, in where indivisibility is needed in order to secure body's substantiality. Kurt Smith and Alan Nelson, 'Divisibility and Cartesian Extension', in *Oxford Studies in Early Modern Philosophy*, vol. v (2010), pp. 1–24.

[29] Here, I am reminded of something Wittgenstein says in *On Certainty*, '141. When we first begin to believe anything, what we believe is not a single proposition, it is a whole system of propositions. (Light dawns gradually over the whole.)' And, '142. It is not single axioms that strike me as obvious, it is a system in which consequences and premises give one another mutual support.' Both are found on page 21e. *On Certainty*, translated by Denis Paul and G. E. M. Anscombe (New York: Harper Torchbooks, 1969).

[30] AT XI 33–4; CSM I 91.

He goes on to say (still in Chapter 6):

> The idea of this matter is included to such an extent in all the ideas that our imagination can form that *you must necessarily conceive it or else you can never imagine anything at all*. Nevertheless, the philosophers are so subtle that they can find difficulties in things which seem extremely clear to other men, and the memory of their 'prime matter', which they know to be rather hard to conceive, may divert them from knowledge of the matter of which I am speaking. Thus I must tell them at this point that, unless I am mistaken, the whole difficulty they face with their matter arises simply from their wanting to distinguish it from its own quantity and from its external extension—that is, from the property it has of occupying space . . . But they should also not find it strange if I suppose that the quantity of the matter I have described does not differ from its substance any more than number differs from the things numbered.[31]

Matter is what makes *physical* (empirical) objects intelligible to us. But, it does not seem to stop here. As Descartes implies in this last passage, matter and quantity and number are, in Cartesian terms, only *rationally* or *conceptually distinct*. Thus, one would be deeply confused if one thought that one could conceive of quantity or number independently of extension (matter). And so, matter seems to also be what makes mathematical objects intelligible to us.

This last claim is supported by something Descartes says in Rule Four of the *Rules*:

> [T]he exclusive concern of mathematics is with questions of order or measure and that it is irrelevant whether the measure in question involves numbers, shapes, stars, sounds, or any other object whatever. This made me realize that there must be a general science which explains all the points that can be raised concerning order and measure irrespective of the subject-matter, and that this science should be termed *mathesis universalis*.[32]

Notice that his list—numbers, shapes, stars, sounds—are all rooted in corporeality. That is, they are ideas the intelligibility of which is underwritten by the idea of matter. Sound looks to be the exception, but as he will discuss the nature of sound later in the *Rules*, it is recast in terms of material objects, strings in this case, which vary in thickness, and which subsequently vibrate at differing frequencies.[33] He is not talking about the *quale* (or sensible quality) sound here. (I might note that his discussion of sound in the *Rules* is connected directly to his discussion of the magnet; so he sees a connection between how the method of discovery should go about its study of both the magnet and sound.) In *The World*, which I quoted above, he again draws the distinction between a sound's being taken to be a 'quality' of experience versus its being 'a certain vibration of air which strikes our ears'.[34] And, this is repeated in later works.[35] Thus, sound understood as a vibration of strings is an idea whose intelligibility is underwritten

[31] AT XI 35; CSM I 91–2, italics mine.

[32] AT X 377; CSM I 19.

[33] AT X 431; CSM I 52.

[34] AT XI 5; CSM I 82.

[35] For example, in the *Optics* (AT VI 131; CSM I 167) and the *Principles* (AT VIIIA 322; CSM I 285).

by the idea of matter. The order (or arrangement) and measure of such things, he says, is the exclusive concern of mathematics. So, matter makes the individuated (empirical) bodies intelligible ('bodies' here being modally divided regions of the plenum), and the arrangeability of these individuated bodies is the sole concern of mathematics.

If the above is right, we can understand why combinatorics became a subject of interest for Descartes, Leibniz, and these others. What we now know is that the mathematical structure of a group, which expresses the very *conditions* of a mathematical system, is found in the arrangeability or combinability of three objects—shown by way of our tricolour group. In fact, we find this holds for as few as two objects (Leibniz's binary 1 and 0 could very well be connected to this insight). The thrust of my thesis is *here*—putting this now in terms of Descartes's insight between matter, divisibility, arrangeability, and mathematics: were God to create at least two things (here, 'things' is broad enough to cover two regions of extension having different shapes), where combination is possible, it follows that God has created the very conditions sufficient for mathematics. In other words, contrary to something like a Neoplatonist view, where mathematics is higher in the hierarchical *chain of being* than is matter, and matter simply *conforms* to mathematics (by divine institution), the view that I am working out here tells us that the two are in some important sense *identical*, which if true, would at the very least make true the conditional stated earlier, namely, *if matter exists, mathematics is intelligible*. For, matter's existing is *sufficient* for establishing the conditions of mathematics. How so? Matter brings with it its (modal) divisibility, and hence arrangeability of that which is divided.

In contrast to a Neoplatonist view, then, God is not taken to have created mathematical truths, and then to have created the two bodies (where the two bodies here can simply be modally distinct divisions in the plenum), and *then* to have ordained that the bodies conform to mathematics. There is no *divorce* between bodies and mathematics here. The insight of Descartes, Leibniz, and the others considered, seems to be that if God creates two or more bodies, the deed is done. For, in creating two or more bodies God *ipso facto* creates a *mathematical* reality. In essence, God creates a permutation group. And, given that *two* is a quantity, the division that produces number, recall, must be a division of a material substrate—for, as Descartes says, quantity is only rationally distinct from matter (extension). If this is right, the mathematical reality is internally (conceptually) tied to matter, divisibility, and combinability.

This is in part an answer to Aristes' question, the sort which prompted our examination, namely, why does God create matter at all? He does so in order to instantiate the very conditions necessary (and sufficient) for mathematics. Of course, a new question arises, which is, 'Why does God care to instantiate mathematics?' But this is another examination altogether (which we will not pursue), and it is one that Descartes rejected, for he held that it was impossible to know the aim of the divine being (here, in Aristotelian terms, Descartes rejects our ever knowing the final cause of the cosmos). Even so, their concern over the creation of matter, I contend, is in part what has got Descartes, Leibniz, and the others, so interested in thinking of synthesis in terms of combinatorics (outside of its applications). Synthesis, understood as the art of combinations, with reference to the permutation group, is a genuine 'mathematized' or mathematical system. And, it is a system expressing a structure that is mathematically richer than a Boolean algebra, the latter constituted of only *monoids*—for, synthesis qua system of permutations is a *group* (recall that a group has inverses, whereas a monoid does not).

A combinatorial system, as construed in the tricolour case, is a permutation group, where the conditions expressed by a group *are* the very conditions underlying *all* of mathematics. And given that the concepts of physics (force and quantity of motion, for example) are constructed by way of synthesis, there is something already inherent in them that is mathematical. In fact, the very ratio scales that define corporeal dimensions are 'mathematical' in this very sense. This in part is what is *conceptually* underwriting what I am calling the 'mathematization' of early modern physics.

◆

Recall in Part II that analysis was cast as a procedure of collection and division. The procedure worked mainly by applying equivalence relations to a set of objects (the objects we wanted to investigate), where the objects were divided by being collected into mutually distinct species-classes, these latter all collected into one genus-class, species and genus forming a hierarchical structure. The species-classes were called *equivalence classes*. One of the examples we considered was a bunch of marbles scattered about on the floor. By applying the equivalence relation '*x has the same colour as y*', we were able to sort each marble into exactly one jar (species-class): blue marbles together in one jar, red marbles together in another, and so on, where the union of the jars gave back the original bunch of marbles. In doing this, we established what was called a *partition* of the marbles, or as Descartes put it, an *enumeration*. There is a connection between our making a partition by way of analysis and the method of synthesis, where (as we cast it above), synthesis is understood in terms of the group concept.

Let's start by reconsidering the finite group, mod-3. Recall that the set S of objects was {0, 1, 2}, and our operator was ⊕. A question that might have been raised when first considering this group is: What about the other integers? That is, what about 4, −5, 98, 403, and so on? Why only 0, 1, and 2? Recall that ⊕ was defined as follows:

$$a \oplus b = \text{ the residue (remainder) of } \frac{(a+b)}{3}$$

If we look back at our table, we will see that $1 \oplus 2 = 0$. What would happen *if* we 'added' 1 and 5? That is, what is the sum of $1 \oplus 5$? Let's see:

$$1 + 5 = 6$$

$$\frac{6}{3} = 2 \text{ (0 is the remainder)}$$

So, our answer is 0. And so,

$$1 \oplus 5 = 0$$

5, in other words, 'acts' just like 2. In fact, 3 will act like 0, 4 like 1, 5 like 2, 6 like 0, 7 like 1, 8 like 2, and so on. This is easily displayed if we took, for example, the integers 0 through 9 and assigned to each 0, 1, or 2 (top row in boldface). This looks as follows:

$$\begin{array}{c} \mathbf{0\ \ 1\ \ 2\ \ 0\ \ 1\ \ 2\ \ 0\ \ 1\ \ 2\ \ 0\ldots} \\ \ldots 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ldots \end{array}$$

The partition extends the 'other' way, for the negative integers, too, though for simplicity's sake I have only shown here the positive integers up to 9. The set S of objects {0, 1, 2} represents a partition of the integers. To emphasize this fact, the set is sometimes written {[0], [1], [2]}, where the bracketed [0], for instance, represents an equivalence class—namely all those integers that act like 0 when $\oplus$ is employed (defined as it was for mod-3 addition). [1] represents the equivalence class of all the integers that act like 1, [2] for those that act like 2. The integer 5, then, is a member of [2]. One way to think of this is that the binary operator $\oplus$ acts as an *equivalence relation* that partitions the integers (here into the classes [0], [1], and [2]). The same holds, of course, if we instead define $\oplus$ in terms of what we called mod-5 or mod-7 addition. Here, for the former the set is {[0], [1], [2], [3], [4]}, and for the latter {[0], [1], [2], [3], [4], [5], [6]}. Mod-5 addition, then, partitions the integers into 5 equivalence classes, mod-7 addition partitions the integers into 7 equivalence classes. What is more, this can be extended to our *ordinary* mathematical operators. They are equivalence relations that partition a set of objects, and in the case of the integers, each integer (even though there is an infinite number) is an equivalence class.[36]

What this shows is that analysis and synthesis go hand in hand, and are internally related (that is, they are *necessarily conceptually* related), such that if we establish the one, we establish the other. Recall from Part I that Russell was noted as saying that a mathematical system can be understood in terms of equivalence relations and classes.[37] Why is a system established by way of an equivalence relation *mathematical*? We now have an answer: every system one establishes by way of analysis is isomorphic to a system established by way of synthesis, and the system established by way of synthesis (which is combinatorial) is a mathematical system. And whether Descartes, Leibniz, Euler (or Plato or Aristotle) actually *knew* this is irrelevant to my point. For, in their attempts at either working out analysis (for this Descartes was primarily considered), or synthesis (for this Descartes and Leibniz were primarily considered), they were, whether they knew it or not, working out the conditions for doing both, where the result of either method is a *mathematical* system. And so, if I am right, it is the development of the methods of analysis and synthesis, as I have laid them out here, which accounts for the conceptual move towards establishing a 'mathematized' physics.

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[36] This, it seems to me, is Frege's insight. See, for instance, his *The Foundations of Arithmetic*, translated by J. L. Austin (Evanston: Northwestern University Press, 1968).

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[37] Bertrand Russell, *The Principles of Mathematics* (Cambridge: Cambridge University Press, 1903), p. 3.

PART IV

SENSIBLE AND INTELLIGIBLE MATTER

13

Is Matter Real?

What does Descartes mean when claiming that matter is *real*? Commentators agree that he held matter to be real in the sense that it exists independent of (finite) mind.[1] No doubt the Sixth Meditation proof for the existence of body was taken by Descartes to establish just this. Even so, there has emerged over the past several decades serious disagreement among commentators as to what *exactly* the Sixth Meditation proof establishes. Along the lines of the traditional reading, some interpret Descartes as establishing that bodies (plural) exist independent of the mind, and although 'they may not all exist in a way that exactly corresponds with my sensory grasp of them,' Descartes says, 'at least they possess all the properties which I clearly and distinctly understand'.[2] Others interpret Descartes as establishing that only one body (identified as *res extensa*) exists independent of the mind. On this reading, Descartes is cast as an idealist with respect to *individuated* bodies. I want to focus on this controversy, for its resolution, which I shall develop in what follows, reveals that Descartes's metaphysical position on the reality of matter is richer than either of these views suggests.

Resolving this controversy will be in service to the principal aim of the book, which is to establish the sense in which Descartes and others held that *mathematics is intelligible if, and only if, matter exists*. The issues to be discussed in Part IV will give me the opportunity to better establish the conditional, *if mathematics is intelligible, matter exists*. For, concerns have been raised at various points in the book about this conditional. It was noted, for instance, that given the possibility of conceiving things that do not exist, extension might be conceived, thus rendering mathematics intelligible, and yet extension (like Pegasus) might not exist independently of thought. In such a scenario we appear to establish the conditions in which the antecedent of the conditional is true ('mathematics is intelligible' is true) and its consequent is false ('matter exists' is false), rendering the conditional false. The reader may recall that a proposed solution was offered back in Part II, but we were not then in a position to work up any details. I promised that we would do that when we reached Part IV. We have arrived.

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[1] Descartes suggests as much in his reply to Caterus, AT VII, 103; CSM II, 75. Lennon claims that Descartes 'never used the term "real" to mean anything but mind-independent'. Thomas M. Lennon, 'The Eleatic Descartes', *Journal of the History of Philosophy*, vol. 45, no. 1 (2007), p. 30. Here, Lennon follows Chappell's rendering of a distinction between *being* and *reality*, a distinction that says that the former is a binary concept (something either depends on the mind or it does not), whereas the latter is a concept that allows for degrees—there can be degrees or levels of reality. 'Real', if I understand Lennon's point, is cast in the light of the former concept.

[2] I have broken up the passage to be more concise. The entire passage is: 'They may not all exist in a way that exactly corresponds with my sensory grasp of them, for in many cases the grasp of the senses is very obscure and confused. But at least they possess all the properties which I clearly and distinctly understand, that is, all those which, viewed in general terms, are comprised within the subject-matter of pure mathematics' (AT VII, 80; CSM II, 55).

It should not go unmentioned that in Part III it was shown how philosophers of the period were in a position to hold the other conditional, *if matter exists, mathematics is intelligible*. Notice that this conditional would be true even if matter did not exist (this is a logical point). Noting this, however, would only satisfy someone interested in one of those negative accounts I mentioned in Chapter 8. In terms of what I called the positive account, we learned that given that matter (extension) is divisible by its very nature (divisibility is guaranteed by instantiating extension), and that any subsequently divided regions are combinable (combinability or arrangeability is guaranteed by division), it follows that if God creates a divided[3] plenum, God instantiates all of the conditions necessary and sufficient for mathematics. God, in essence, creates a permutation group. The instantiation of the conditions of mathematics is coextensive with its intelligibility. So, the positive account tells us that if matter exists, instantiating these conditions, mathematics is intelligible.

The resolution of the controversy between the realist and idealist readings of Descartes needs some stage setting. To do this, we will reconsider Plato's divided line analogy, discussed in Part I, our inquiry coming full circle. Here, I shall appeal to Kant for help. As we shall see, Kant's empirical/transcendental distinction aligns nicely with Plato's Visible/Intelligible distinction. Kant's way of putting these distinctions will be of great help in our working out the senses in which Descartes took matter to be real. I shall rely heavily on Henry Allison's reading of Kant. So, let me set the stage.

THE DIVIDED LINE RECONSIDERED

Henry Allison develops two versions of the Kantian distinction between appearances and things-in-themselves—an *empirical* version and a *transcendental* version.[4] I will adopt Allison's view here. The two versions of the distinction are built out of the concepts of *ideality* and *reality*, which are important concepts for Kant.[5] Starting with the empirical version, Allison says:

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[3] Division here need be only conceptual or modal—it need not be *real*.

[4] Henry Allison, *Kant's Transcendental Idealism* (New Haven: Yale University Press, 1983). Allison does not make the connection between Plato and Kant that I make. So, any problem the reader sees with the connection should be attributed to my use of Allison's reading and not to Allison.

> Taken in the empirical sense, 'ideality' characterizes the private data of an individual mind . . . 'Reality', construed in the empirical sense, refers to the intersubjectively accessible, spatiotemporally ordered realm of objects of human experience. At the empirical level, then, the ideality-reality distinction is essentially between the subjective and the objective aspects of human experience.[6]

That which is *empirically ideal* is that which depends in some important sense on a particular mind. In this case it cannot exist independently of this mind's act of perception. By contrast, that which is *empirically real* is that which is, in an equally important sense, independent of any particular mind. Such a thing can exist regardless of whether this or that mind perceives it.

Although Allison does not mention it, one cannot help but be struck by similarities between his discussion of the *empirical ideality-reality distinction* and a discussion of G. E. Moore's found in his famous paper, 'Proof of an External World.'[7] There, Moore makes a similar point regarding a thing's being *internal* and *external* to a mind in terms of *dependence* and *independence* of minds, respectively.[8] In his analysis, Moore focuses on two phrases used by Kant: 'to be presented in space' and 'to be met with in space'.[9] He applies the phrases in an examination of several cases. I will begin with the case of an after-image.

Suppose that I stare at the sun for several seconds and then look away. What I will 'see' is a largish, round, yellow-white object, hovering 'out there' at what appears to be some distance from me. This object, the after-image of the sun, is, as Moore says, 'presented in space'. That is, it appears to me *as though it were a spatial object*, located some distance from me. As I said, it is presented (this is all phenomenological talk) 'out there' in space. Even when I close my eyes, the object appears to be presented as though it is located at a 'place' some distance from 'me', the perceiving subject. The after-image exists as long as I perceive it. Once the image fades completely, the object no longer exists. At least, this is what we are prepared to believe about it. Its existence depends in an important sense on *my* mind. Not so, of course, with the sun.

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[5] Kant introduces the concepts in § 3: 'The Transcendental Exposition of the Concept of Space', in *Critique of Pure Reason*, translated by Norman Kemp Smith (New York: St. Martin's Press, 1965), pp. 72ff.

[6] Allison (1983), pp. 6–7.

[7] G. E. Moore, 'Proof of an External World', in *Philosophical Papers*, Chapter VII (London: George Allen & Unwin Ltd, 1959), pp. 127–50. Joseph Schear insightfully noted, in an exchange we had during my visit to Calpoly San Luis Obispo, that one reason that Allison may have ignored Moore's handling of the distinction is Moore's sole emphasis on *space*—where, as we will see, Moore focuses on the phrases 'to be presented in space' and 'to be met with in space' in order to explicate Kant's distinction between empirical ideality versus empirical reality. As Schear pointed out, Kant claims that *time*, as a condition of experience, could stand despite the annihilation of the condition of space (or the 'vanishing' of appearance). See Kant's *Critique of Pure Reason*, The Transcendental Aesthetic, section II, Metaphysical Exposition of the Concept of Time, A31 (Norman Kemp Smith's translation, p. 75). Being as careful as possible, then, I will nevertheless use Moore's explanatory machinery along with Allison's.

[8] Moore (1959), pp. 141–4. Here Moore says that the independence is 'logical' in nature (p. 144).

[9] Moore seems to be looking at the 'Fourth Paralogism' of *Critique of Pure Reason* (Norman Kemp Smith, pp. 334–52).

The sun is also 'presented in space'—that is, when I look at it. One difference, at least according to Moore, is that the sun, contrary to what we think about the after-image, exists whether I look at it or not. In other words, the sun does not depend for its existence on my mind (or, at least this is what we are prepared to believe about it). When I am not looking at it, the sun is nevertheless taken to exist. In fact, the locution 'looking at' is revealing, for we do not say that one looks *at* an after-image—there is simply nothing at which to look (this is different from saying that one 'pays attention' to the after-image). Rather, the after-image is an artefact of the very *act* of perceiving. Properly speaking, it is not an object *of* perception. It is an artefact being produced internal to the processes of perception, an artefact that when being produced disturbs my ability to look at other things. This sort of linguistic difference is not the only difference between what we believe about the sun and my after-image of it—and, perhaps, it is not among the more interesting differences. One significant difference is our belief that the sun that I look at is the *very same* sun that you look at. And so, if we both stare at the sun a bit too long, we would count *two* after-images (yours and mine) but only *one* sun. The latter sort of object is not only 'presented in space' (when someone perceives it), but is 'to be met with in space'. This last phrase is supposed to capture the two additional beliefs had about such objects as the sun, which express the 'mind-independence' feature, namely

1) Such objects can exist whether perceived or not,

and,

2) Two or more perceivers can perceive such objects.

Thus, when perceived, the after-image is presented in space, but because it cannot exist independently of my perceiving it, and you cannot ever perceive my after-image, it is not the sort of thing that is to be met with in space. Contrast this with the sun. When perceived, the sun is not only presented in space, but because it can exist independently of its being perceived, and can be perceived by multiple perceivers, it is also to be met with in space. It is important to note that Moore does not take himself to be answering a question of *empirical science* here—that is, he is not asking, 'Is there really a star at the centre of our solar system?' Rather, the question is a philosophical one, which is more akin to the sort asked by Berkeley, namely, 'What is conceptually involved in our thinking that the sun exists when not perceived?' Or, perhaps better, 'What is involved in our conceiving of a mind-independent world?' Clearly, at the level of empirical science, such concerns must already be settled (or ignored). Moore's point is a conceptual or logical one. The difference between subjective, mind-dependent sensory content (that is, what we might call purely *mental* content) and objective, mind-independent objects is for him analysed in terms of what we are prepared to *say* about them (and how we are prepared to act with respect to them). In the case of the latter sort of object, that is, mind-independent objects, we take them to meet the conditions expressed by those two additional beliefs, which capture the meaning of the phrase 'to be met with in space'.

It should be clear, Moore argues, that not everything that is presented in space is to be met with in space, and vice versa. The after-image is presented in space but is not to be met with in space. When the sun is not being perceived by anyone, it is not presented in space but it is for all that to be met with in space. Moore notes that Kant seems to have taken the two phrases to be coextensive, but he takes his own analysis of these phrases to show that they are not.

Since Moore is interested in working out something that would look like a proof of the *external* world, he goes on to examine two senses of 'external to the mind' used by Kant, which are (not coincidentally) *empirical* and *transcendental* senses. An object that is *empirically external* to the mind (like the sun) is one that is to be met with in space, and when perceived, presented in space. An empirically external object is an empirically real object. On the other hand, an object that is *empirically internal* to the mind (like the after-image) is one that is never to be met with in space, and is in fact solely presented in space. Empirically internal objects are empirically ideal. This, I will suggest, is what we have with respect to Allison's empirical senses of 'ideality' and 'reality'. An object that is *empirically internal* (to the mind) in Moore's sense is *empirically ideal* in Allison's sense, and an object that is *empirically external* (to the mind) in Moore's sense is *empirically real* in Allison's.

Concerning the transcendental version of the distinction, Allison says:

> The transcendental version of the distinction is quite another matter. At the transcendental level, which is the level of philosophical reflection upon experience (transcendental reflection), 'ideality' is used to characterize the universal, necessary, and therefore, a priori conditions of human knowledge. In the Transcendental Aesthetic, Kant affirms the transcendental ideality of space and time on the grounds that they function as a priori conditions of human sensibility, that is, as subjective conditions in terms of which alone the human mind is capable of receiving the data for thought or experience . . . Things in space and time (empirical objects) are ideal in the same sense because they cannot be experienced or described independently of these sensible conditions. Correlatively, something is real in the transcendental sense if and only if it can be characterized and referred to independently of any appeal to these same sensible conditions . . . A transcendentally real object is thus, by definition, a nonsensible object or noumenon.[10]

In the transcendental version of the distinction, 'ideality' refers to objects understood in terms of the *a priori* conditions that underwrite their being intuited

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[10] Allison (1983), p. 7.

(or perceived). Interestingly enough, these *objects* are none other than those found in the previous empirical version of the distinction, which we called *empirically real* objects (material objects such as the sun). When we think of empirically real objects in respect to what is necessary for their being intuited or experienced at all (that is, being thought as empirically real), we are working in the category of the transcendentally ideal. Transcendental ideality marks the very conditions necessary for all human knowledge and experience of the empirically real (and empirically ideal for that matter). By contrast, transcendental reality marks off that which is not subject to these conditions—specifically, transcendentally real objects are not subject to the *a priori* conditions of space and time. Since these latter conditions are necessary for our having any experience (or even conception) of an empirical object, it will follow that we cannot think about (or know anything about) objects that would count as being transcendentally real. Such objects are, as Kant famously calls them, *noumena*.

Moore, recall, noted that Kant held there to being empirical and transcendental senses of 'external to the mind'. Empirically real objects, as was noted above, are empirically external to the mind; and, it should be no surprise, transcendentally real objects are transcendentally external to the mind. Since the latter sort of object cannot be conceived, a proof about them might be thought to be pointless—for, *what* is it about which one is claiming to make a proof? On Moore's view, the only external objects for which a proof is possible are those that are empirically external to the mind. Consider the following table, then:

Table 13.1

	Ideality	Reality
Empirical	A	B
Transcendental	C	D

A-type objects are empirically ideal. The after-image of the sun is an example of an A-type object. B-type objects are empirically real. The sun itself, existing in the heavens, is an example of a B-type object. B-type objects are empirically external to the mind, and this is the world for which Moore makes his famous proof.[11] Now, when we take B-type objects and consider *them* in light of what is necessary for their being conceived at all (which in part involves us in asking what it is that makes them distinct from A-type objects), we recast them as C-type objects. That is, we cast them in terms of the conditions of conceivability. C-type objects are transcendentally ideal. It might be stressed here that given solely A-type objects, Kant finds it impossible to account for our ever arriving at the conception of B-type objects—for the latter cannot be *built* out of the former. On this view, the conditions discovered at the level of transcendental ideality underwrite, or account for, the very possibility of the distinction made between A and B-type objects. Finally, D-type objects are transcendentally real, and according to Moore's analysis are transcendentally external to the mind. *What* these objects are, of course, is impossible to say, since they are not subject to the conditions of space and time (and so, are not conceivable).

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[11] Though it is worth noting that Moore wonders out loud, so to speak, why a proof for such a thing ever occurred to folks, and why philosophers were ever led to think that they needed to provide such a proof. Moore's analysis shows that the existence of the empirically external world is already logically or conceptually *embedded* in our ways of talking. Raising a concern about the existence of the empirically external world runs close to a kind of mental disorder—this is the sort of insight that Wittgenstein found valuable in Moore's essay, when he took to writing *On Certainty*. The proof, on Moore's view, amounts to simply showing how a statement like 'Here are

The above categories are useful on several fronts. For instance, we could understand philosophical systems in light of them. Someone who took the epistemically basic objects to fall into the category of empirical ideality, for example, could be referred to as an empirical idealist. Someone who took the epistemically basic objects to fall instead into the category of empirical reality could be referred to as an empirical realist, and so on. Kant appears to think that Berkeley, for example, took the epistemically basic objects to be those falling into the category of empirical ideality. Hence, for Kant, Berkeley was an empirical idealist.[12] Kant takes himself to be a transcendental idealist, holding that the ground and limit of human knowledge (about the world) is ultimately to be found in the category of transcendental ideality.

Allison argues that in terms of the *empirical ideality-reality distinction*, which he notes is included in what Kant calls the 'language of experience',[13] 'appearance' refers to the empirically ideal whereas 'things-in-themselves' refers to the empirically real. Here, the latter's being taken to exist independently of being perceived (that is, independently of the subjective instances of perception) constitutes, in part, their being taken in this context as things-in-themselves. By contrast, in terms of the *transcendental ideality-reality distinction*, 'appearance' refers to the transcendentally ideal, whereas 'things-in-themselves' refers to the transcendentally real. Since C-type objects are B-type objects *recast* in light of the conditions of the conceivability of B-type objects, we can understand why

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two hands' logically or conceptually entails that there are at least two empirically real objects. The *proof*, in other words, is really a philosophical clarification of the logic (or conceptual landscape) of the language.

[12] Of course, there is the *Second Dialogue*, in which Philonous introduces the notion of the Omniperceiver. Here, the Omniperceiver accounts for the existence of such things as tables, chairs, and the sun (empirically real objects) when finite minds are not perceiving them. That is, such objects are said to exist (or *could* exist) independently of a human mind. And so, arguably, with the introduction of the Omniperceiver, Berkeley could be taken to be a strange kind of empirical realist.

[13] Allison (1983), p. 8.

it is that Kant sometimes will refer to what, in the empirical version of the distinction, are empirically real objects now as *appearances* or *phenomena*. But when doing so, Allison contends, Kant is taking these objects not in the light of the empirical version of the distinction (as B-type objects) but in light of the transcendental version (as C-type).[14] In working out the above versions of the distinction, then, Allison shows that Kant is not being inconsistent when sometimes referring to these objects *qua* empirically real as things-in-themselves and to the very same objects, though this time *qua* transcendentally ideal, as appearances or phenomena.

Let me suggest the following. There is a similarity between the four sections of Plato's divided line and the four categories of Kant. The Visible (or Sensible) division of the line, I contend, aligns with Kant's empirical version of the ideality-reality distinction: Plato's line section representing images is similar to Kant's category of the empirically ideal; Plato's line section representing objects *qua* originals is similar to Kant's category of the empirically real. What is more, the Intelligible section aligns with Kant's transcendental version of the ideality-reality distinction: Plato's line section representing objects *qua* images is similar to Kant's category of the transcendentally ideal; Plato's line section representing the Forms is similar to Kant's category of the transcendentally real.

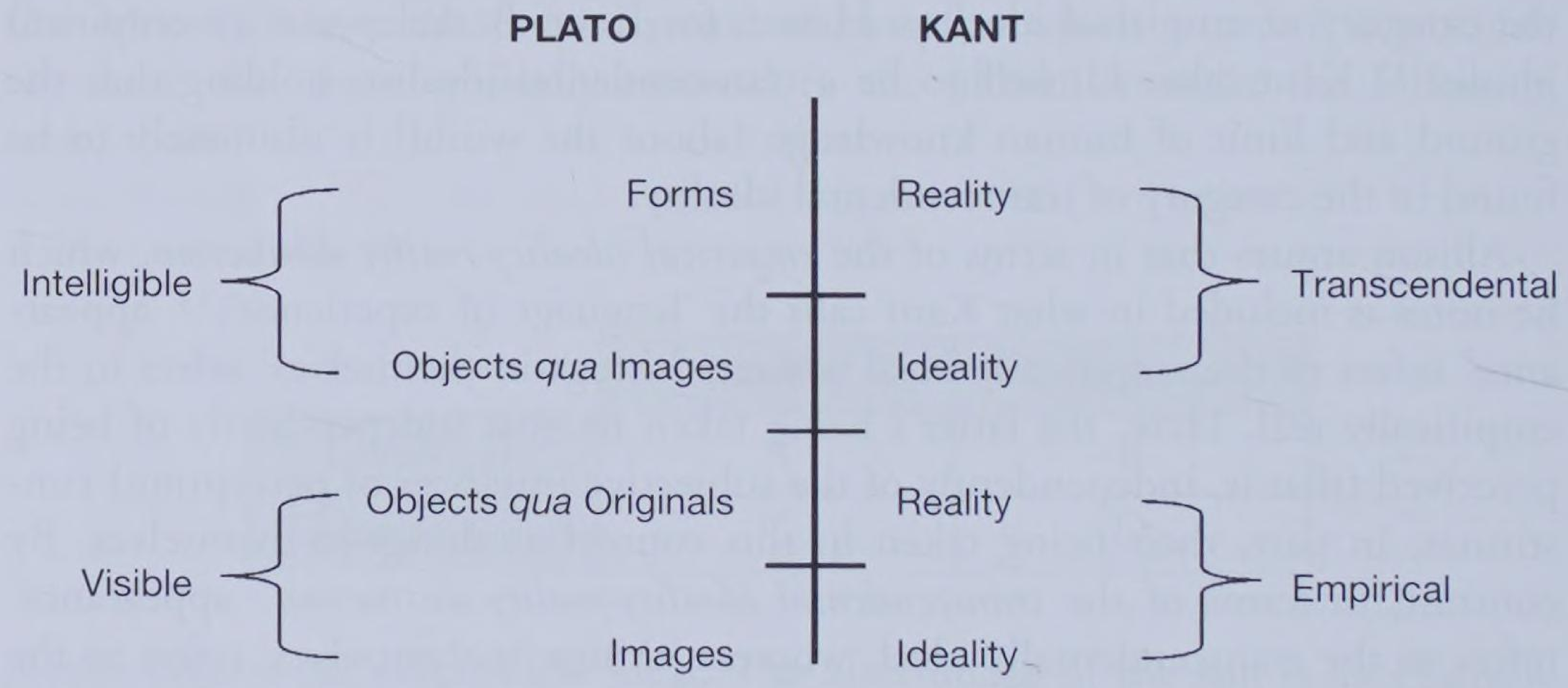

Figure 13.1

To be sure, there will be important differences here,[15] but the similarities are enough for us to make clearer Descartes's position on the extra-mental status of matter. I will first examine the sense in which Descartes took body to be empirically real, which I think is the traditional reading of his view, and then go on in the following chapters to examine the two possible senses in which he took body to be either transcendentally ideal or transcendentally real. I shall use these distinctions to resolve the controversy between those realist and idealist readings mentioned earlier.

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[14] Allison (1983).

[15] For instance, Kant does not obviously buy the claim that the Forms (or Ideas) are the transcendentally real objects. Even so, like Plato, he holds that whatever these items turn out to be, they are independent of the conditions of space and time (in other words, they are *eternal*).

14

Empirical Ideality, Reality, and Matter

In an appendix to the Fifth Replies, Descartes addresses Clerselier, highlighting criticisms that Gassendi, author of the Fifth Objections, had made concerning an extra-mental, material world. Descartes refers to it as 'the objection of objections',[1] which goes as follows:

> They say that many people of great intelligence think they clearly see that mathematical extension, which I lay down as the fundamental principle of my physics, is nothing other than my thought, and hence that it does not and cannot have any subsistence outside my mind, being merely an abstraction which I form from physical bodies. And they conclude that the whole of my physics 'must be imaginary and fictitious, as indeed the whole of pure mathematics is, whereas real physics dealing with the things created by God requires the kind of matter that is real, solid and not imaginary'.[2]

Descartes, in effect, understands his objectors to take his extension to be a purely intelligible (that is, ideal) matter, which cannot exist independently of the mind. They appear to contrast his extension to sensible matter. Unlike intelligible matter, sensible matter is impenetrable, has weight, and so on, which can (and presumably does) exist independent of the mind. Aquinas, recall, held a similar distinction, which, as I suggested in Part I, seems to be a distinction made even earlier by Plotinus. Aquinas claims that mathematics took intelligible matter as its object, whereas physics took sensible matter as its object. The trouble seems to be that intelligible matter does not guarantee the existence of sensible matter. And so, if all Descartes can establish in his metaphysics is intelligible matter, he will be unable to secure the claim that there exists an extra-mental, material world. Since he cannot secure this, his physics, in dealing solely with intelligible matter, will not tell us anything about a mind-independent material world.

Friedman has taken on the explicit task of working out Descartes's Sixth Meditation proof for the extra-mental existence of matter.[3] There are several excellent studies of this proof in the secondary literature,[4] but I want to focus on Friedman's study because of the connections he makes to Kant. The details of the proof, as worked out by scholars, will be of little importance here. My interest is in what Descartes took the proof to establish. For, this will reveal how he conceived the problem of matter, and how he took his response to be a solution to that problem—we can look at this regardless of whether his solution ever actually succeeded.

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[1] AT VII 212; CSM II 275.

[2] AT VII 212; CSM II 274–5.

[3] Friedman (1997), pp. 153–62.

[4] Three that come immediately to mind are Anthony Kenny's *Descartes: A Study of His Philosophy* (New York: Random House, 1968), Margaret Wilson's *Descartes* (London: Routledge & Kegan Paul, 1978); and Richard Field's short article, 'Descartes' Proof of the Existence of Matter', *Mind* (1985) XCIV (374), pp. 244–9.

The challenge for Descartes, Friedman says, is to show how his matter does not have what Friedman calls 'ideal Platonic existence', but in fact has what he calls 'real or actual existence'.[5] As I understand Friedman's characterization here, and in light of what Descartes says his objectors claim, the former sort of existence is that which is connected to its being simply the subjective mental content of a mind, and the latter sort is that which is connected to its being an extra-mental object—a thing that exists independent of the mind. Although I can adopt much of Friedman's account, I disagree with his characterizing the 'subjective'[6] content of ideas in terms of the *ideal* as that term is connected to Plato. For Plato the *idea* or *eidos* (Form) was a *substance* of the cosmos. It was not 'ideal' in the sense that it was subjective, purely mental content. Neither was it an abstract idea, in the sense that one gets the idea as a result of focusing on one element in it while ignoring others. For Plato, ideas were the *most* real things in the cosmos, everything else ontologically and epistemologically depending on them.

In light of Plato's divided line, here emphasizing the *mind-dependent* versus *mind-independent* distinction, the Sixth Meditation proof for the existence of body (at this stage I am not considering whether there is only one or a plurality of bodies) should be seen as working within the Visible or Sensible section of the line, Friedman casting the proof as one that establishes body as something in the category of objects *qua* originals. Or, in Kant's terms, the proof works within the limits of the *empirical ideality-reality distinction*. And so, what the proof establishes is that extension is empirically real. If this is right, the items in the lowest category, images, are not *Platonic forms* or *ideas* (those, according to Plato, are located in the Intelligible section, in the highest category of the divided line). And so, as I said above, the transition is not a move from an ideal Platonic existence to a real or actual existence—for that would presumably move us from the top category of the Intelligible section *down* to the top category of the Visible section—but rather is a move from the category of subjective mental content (the mind-dependent) to the category of extra-mental existent (the mind-independent). Within the context of the Sixth Meditation, the proof looks to be best understood as moving from the bottom category of the Visible section *up* to the top category of the Visible section.

In line with how Descartes describes the challenge of his objectors, and specifically with his emphasizing *sensation* in the Sixth Meditation proof, the transition is a move from a category of object that is understood to be utterly dependent on an individual mind (a particular instance of sensory perception), to a category of object that is independent of *that* mind (or any particular instance of sensory perception). As I said, in proving matter exists, Descartes makes the conceptual transition from the empirically ideal to the empirically real; and so, to say that matter *really* exists, as Friedman puts it, is to say that it is empirically real. This is what the proof establishes.

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[5] Friedman (1997), p. 155.

[6] Ibid., p. 159.

Friedman describes the gist of the Sixth Meditation argument as follows:

> To say that purely geometrical extended quantity has real as opposed to merely Platonic existence, then, is to say that purely geometrical extended quantity alone—not God, not myself, and not some other created non-extended substance—is the cause of my sensible ideas.[7]

As I noted, we might ignore the reference to *Platonic existence* here, and instead read Friedman as saying that the aim of the Sixth Meditation proof, in turning on the claim that extension is the *cause* of sensory experience, is to establish that extension (or matter) is empirically real. Recall from Part II that ruling out that God, one's own mind, or some other mind (like that of an angel) is the cause of the idea of body was done by way of the enumeration. Since that part of the proof has already been fully discussed, we need not go into any details here. So, I will move forward.

'The argument for real existence,' Friedman adds, 'is completed . . . by using God's veracity.'[8] It is well known that by the close of the Third Meditation Descartes establishes that his nature is directly the result of God's handiwork.[9] What is more, God is responsible for giving Descartes the impulse or great propensity to believe that his sensations are caused by extra-mental objects, and in particular, extended things (bodies). Conjoined with the Fourth Meditation's proof that God cannot be a deceiver, Descartes argues:

> So I do not see how God could be understood to be anything but a deceiver if the ideas were transmitted from a source other than corporeal things. It follows that corporeal things exist.[10]

This part of the proof, as Friedman notes, appeals to the veracity of God. Ultimately, the proof shows that not only do bodies exist, but lends itself to showing that 'they possess all the properties which I clearly and distinctly understand, that is, all those which, viewed in general terms, are comprised within the subject-matter of pure mathematics'.[11] The list will include shape, size, motion, and position.

Descartes takes 'external to the mind' and 'independent of the mind' to be coextensive.[12] And, he takes 'independent of a finite mind' to be coextensive with 'real'.[13] In proving that matter exists external to the mind one *ipso facto* proves that it is real. Within the context of the *empirical ideality-reality distinction,* to be independent of the mind would at the very least mean that body exists independently of the mind's subjective mental content—it would exist independently of any particular moment or instance of perception. This, as I said, would place body in the category of empirical reality. So, the Sixth Meditation proof proves that matter is empirically real.

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[7] Ibid., p. 156.
[8] Ibid., p. 157.
[9] AT VII 52; CSM II 35.
[10] AT VII 80; CSM II 55.
[11] Ibid.
[12] Moore (1959), p. 129ff.; Lennon (2007).

Now, Descartes never used the phrase 'empirically real', so how does he express this concept? What philosophical apparatus does he employ to articulate this category, and the results of his proof? The apparatus is a distinction he introduces in the Third Meditation—*the formal-objective reality distinction.* This distinction goes as follows. There are two kinds of 'reality'—formal reality and objective reality. The formal reality of a thing is the reality the thing possesses in virtue of its being an existent thing. This, I take it, is the concept that Lennon has in mind when connecting *real* with being *mind-independent*.[14] There are at least three levels of formal reality—that of an infinite substance, a finite substance, and a mode. The level of formal reality of that of an infinite substance is greater than that of a finite substance, and that of a finite substance is greater than that of a mode. When Descartes says that he is an existent finite mind, he is saying that his mind possesses formal reality, which is the level of that of a finite substance. An occurring idea is something that possesses formal reality (insofar as it is an existent), which is the level of that of a mode. In the case of all ideas, 'the nature of an idea,' Descartes says, 'is such that of itself it requires no formal reality except what it derives from my thought, of which it is a mode'.[15] So, the formal reality of an idea will have its origin in the formal reality of the mind, of which it is a mode. This is again suggested when he says that insofar as ideas are '[considered] simply [as] modes of thought, there is no recognizable inequality among them: they appear to come from within me in the same fashion'.[16]

By contrast, the objective reality of a thing is the reality it possesses in virtue of its being a representation of something. On Descartes's view, ideas appear to be the only 'things' that possess this sort of reality. Now, in principle the origin of the objective reality of an idea can be other than the formal reality of his mind. Descartes says:

> And although the reality which I am considering in my ideas is merely objective reality, I must not on that account suppose that the same reality need not exist formally in the causes of my ideas, but that it is enough for it to be present in them objectively. For just as the objective mode of being belongs to ideas by their very nature, so the formal mode of being belongs to the causes of ideas—or at least the first and most important ones—by their very nature. And although one idea may perhaps originate

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[13] Lennon (2007), p. 30.
[14] Ibid.
[15] AT VII 41; CSM II 28.
[16] ATVII 40; CSM II 27–8.

> from another, there cannot be an infinite regress here; eventually one must reach a primary idea, the cause of which will be like an archetype which contains formally [and in fact] all the reality [or perfection] which is present only objectively [or representatively] in the idea.[17]

He says, since 'different ideas [are considered as images which] represent different things, it is clear that they differ widely'.[18] This is in contrast to considering his ideas in terms of their formal reality, which as we know tells us that there is no recognizable difference among them: each is a mode of his mind. But in terms of their objective reality—that is, their representational content—the ideas appear to differ widely. So, although there will be no difference between his idea of God, his idea of a man, his idea of the sun, and his idea of the sun's shape, when considering their formal reality (each possesses the formal reality of that of a mode), they nevertheless will differ widely when considering them objectively.

According to the traditional reading, in cases of one's perceiving the extra-mental world, ideas are taken to be those 'objects' that represent to the mind objects in the extra-mental reality. Ideas are the immediate objects of perception, whereas the extra-mental objects (those objects being represented to the mind by way of our ideas) are the mediate objects of perception. Consider the following illustration:

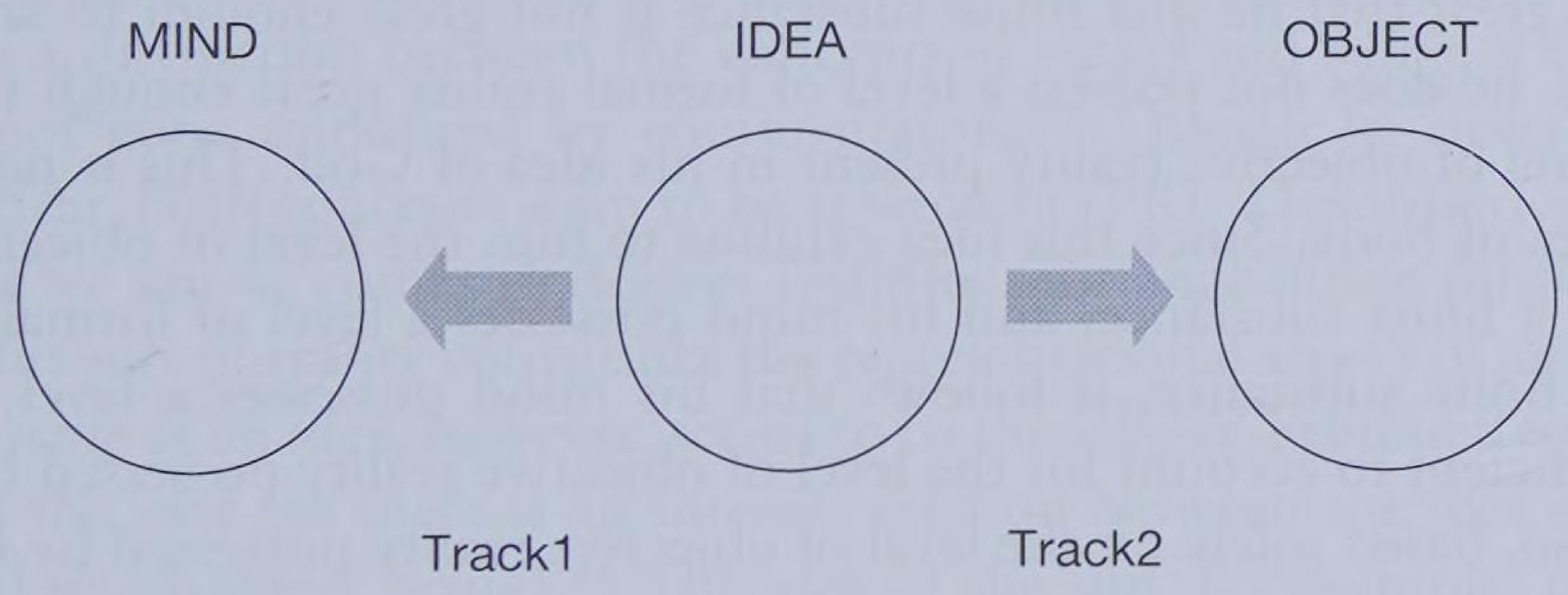

Figure 14.1

As developed in the Third Meditation (this will be within the context of Descartes's theory of ideas), the formal-objective reality distinction tracks the origin of the two kinds of reality an idea possesses—namely, an idea's formal and objective reality. Track1, as I call it in the above illustration (Figure 14.1), tracks the formal reality of the idea to the formal reality of the finite mind (of which it is a mode). Taking the idea *formally* within the context of the distinction sets one on Track1. Track2 tracks the objective reality of the idea to the formal reality of some object, which, when not one's own mind, will be an extra-mental object. This is so, Descartes says, because 'in order for a given idea to contain such and such objective reality, it must surely derive it from some cause which contains at least as much formal reality as there is objective reality in the idea'.[19] Taking the idea *objectively* within the context of the distinction sets one on Track2.

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[17] AT VII 42; CSM II 29.

[18] Ibid.

As I understand this, Track2 is used by Descartes to account for the idea's *representing* the object that it does. I shall call this the *Principle of Representation* (PR).[20] This holds for primary ideas. The principle can be stated thus:

> (**PR**) An idea represents *x* only if the idea's objective reality has its origin in the formal reality of *x*.[21]

The paradigm example of a primary idea is the idea of God. An application of the formal-objective reality distinction yields the following result. Track1 tells us that the formal reality of the idea derives from Descartes's mind, of which it is a mode. Track2 tells us that the objective reality of the idea derives from the formal reality of God. (PR) and the formal-objective reality distinction play important roles in the Third Meditation proof for God's existence. It is no coincidence that (PR) and the distinction also play important roles in the Sixth Meditation proof for the existence of body.

Even so, the Sixth Meditation proof for the existence of body is importantly different from the Third Meditation proof for the existence of God. In the latter, the proof hinges on Descartes's noticing that the *level* of objective reality in the idea is so great that he *qua* finite substance is not great enough to account for it. That is, he does not possess a level of formal reality great enough to account for the level of objective reality present in his idea of God. This is not the case for his idea of body. Since this idea exhibits to him the level of objective reality of that of a finite substance, and his mind possesses a level of formal reality of that of a finite substance, it follows that his mind possesses a level of formal reality sufficient to account for the level of objective reality possessed by his idea of body. So, based solely on the level of objective reality possessed by his idea of body, he cannot prove that body, and not his mind, is the cause of the objective reality present in his idea of body—he cannot prove that body exists (that it possesses formal reality) independent of his mind.

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[19] AT VII, 41; CSM II, 28–9.

[20] I discuss this in more detail in Smith ('Rationalism and Representation', 2005) and Smith ('Descartes's Ontology of Sensation' 2005). For an interpretation that would reject the one on offer here, see Frederick J. O'Toole's 'Descartes' Problematic Causal Principle', in *Descartes's* Meditations, edited by Vere Chappell (Lanham: Rowman & Littlefield Publishers, 1997), pp. 103–28. The trouble that I see for O'Toole's reading is that he wrongly grounds what he calls the Causal Principle of Ideas (CPI) in the material-objective distinction, introduced in the 'Preface of the Reader' of the *Meditations*. I say 'wrongly' since I take CPI to be grounded in the formal-objective reality distinction, and I take the latter distinction to be importantly different from the material-objective distinction.

[21] Phrasing the relation in terms of *origin* is deliberate. I am trying to remain neutral with respect to the interpretative battle over what Descartes means by *cause*, and in particular what he meant when casting an object's formal reality as the cause of an idea's objective reality. Chappell claims that representation is a three-termed relation. By this I believe he means: idea *x* represents object *y* to mind *z*. Vere Chappell, 'The Theory of Ideas', in *Essays on Descartes'* Meditations', edited by Amélie Oksenberg Rorty (Berkeley: University of California Press, 1986), p. 191. I take (PR) to be compatible with Chappell's insight.

As early as in the Second Meditation, Descartes expresses concerns over the elements exhibited to him in his idea of body—elements that in the *Rules*, recall, were called simple natures. The elements include extension, size, shape, and so on. With his Second Meditation remarks in mind, Descartes says in the Third Meditation:

> As for all the other elements which make up the ideas of corporeal things, namely extension, shape, position and movement, these are not formally contained in me, since I am nothing but a thinking thing; but since they are merely modes of a substance, and I am a substance, it seems possible that they are contained in me eminently.[22]

So, the *level* of objective reality is not an issue (for presumably he can account for that), but the presence of *extension* and *shape* that he finds presented by (or in) his idea *is* an issue. How will he account for them? Within the context of the Third Meditation, as we just saw, one way might be to appeal to a well-known scholastic distinction between *formal* and *eminent* possession. But, the appeal is a false start, and Descartes is able to reject it upon his completing the Sixth Meditation proof for the existence of body.

The difference between the Third Meditation proof for God's existence and the Sixth Meditation proof for the existence of body suggests that Descartes is relying on a distinction between the concepts of *reality* and *being*, a distinction that has not gone unnoticed by commentators.[23] Though he never makes it perfectly clear, both concepts seem to be at work in (PR). The objective *reality* of an idea, as we know, comes in degrees (infinite substance, finite substance, and mode). This sort of reality constitutes the representational aspect of an idea. The objective *being* of an idea, however, seems to be the 'object' exhibited to the mind by way of the idea (so there is an internal relation between the idea's objective reality and its objective being).[24] The idea of the sun, for example, possesses a level of objective reality of that of a finite substance. The objective being, on the other hand, is the sun as it is exhibited by (or in) the idea.[25] So, it would appear that in the Sixth Meditation proof for the existence of body, Descartes focuses on the objective being presented by (or in) his idea of body—for example, an extended being that is shaped, sized, and so on. The proof does not turn on an examination of the (level of) objective reality possessed by the idea.

We get some discussion of a distinction between *reality* and *being*, though not exactly in these terms, and a connection to the formal-objective reality distinction, in the First Replies. Although I do not want to dwell on this, it is important enough to warrant a brief look.

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[22] AT VII 44; CSM II 30–1.

[23] See, for example, Chappell (1986), p. 190ff; and Lennon (2007).

[24] Chappell (1986) initially says that *being* is an all-or-nothing affair—either something is or it is not. This makes *being* look like *exists*. But, as the discussion unfolds, he seems to mean by *being*, in the phrase *objective being*, that which is presented to the mind by way of an idea.

[25] I follow commentators such as Chappell (1986).

Caterus, author of the First Objections, finds trouble with Descartes's usage of standard scholastic terminology, which includes Descartes's use of 'objective reality' and 'objective being'. As Caterus understands it, an idea is 'the thing that is thought of, in so far as it has objective being in the intellect'.[26] He asks: what is meant by the phrase 'objective being in the intellect'? He answers: 'According to what I was taught, this is simply the determination of an act of the intellect by means of an object. And this is merely an extraneous label which adds nothing to the thing itself. Just as "being seen" is nothing other than an act of vision attributable to myself, so "being thought of", or having objective being in the intellect, is simply a thought of the mind which stops and terminates in the mind.'[27] So, if the sun, for example, determines an act of the intellect, so that we say (as a consequence) that one is aware of the sun, the sun is said to have objective being in the intellect. This, Caterus says, adds nothing to the sun. For, my seeing the sun is a fact about *me*, not about the sun. My seeing it, in other words, changes nothing about the sun. In this sense, Caterus says, when we say that the sun has objective being (it is an object of thought), the phrase *objective being* is 'an extraneous label which adds nothing to the thing itself'. Because of this, it would be a mistake to think that the sun's objective being requires a cause. For, *objective being* is not a *real* entity—that is, it does not possess formal reality—but is a philosophical (or a linguistic) invention.

Descartes's reply to Caterus on this issue is important. Let me quote the entirety of the relevant portion of the passage before commenting. Descartes says:

> He [Caterus] says, first of all, that when a thing exists in the intellect by means of an idea, it is not an actual entity, that is, it is not a being located outside the intellect; and this is quite true. Next he goes on to say that 'it is not something fictitious or a conceptual entity but something real which is distinctly conceived'; here he concedes everything which I have assumed. But he then adds 'since it is merely conceived and is not actual'—that is, since it is merely an idea and not a thing located outside the intellect—'although it can be conceived it cannot in any way be caused.' This is to say that it does not require a cause enabling it to exist outside the intellect. This I accept; but it surely needs a cause enabling it to be conceived, which is the sole point at issue. Thus if someone possesses in his intellect the idea of a machine of a highly intricate design, it is perfectly fair to ask what is the cause of this idea. And it will not be an adequate reply to say that the idea is not anything outside the intellect and hence that it cannot be caused but can merely be conceived. For the precise question being raised is what is the cause of its being conceived. Nor will it suffice to say that the intellect itself is the cause of the idea, in so far as it is the cause of its own operations; for what is at issue is not this, but the cause of the objective intricacy which is in the idea.[28]

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[26] AT VII, 92; CSM II, 66.

[27] AT VII, 92; CSM II, 66–7.

[28] AT VII, 103; CSM II, 75.

In the Third Meditation Descartes had argued that, although the objective being in an idea will not be found outside a mind (and so, it is not an *actual* being—a being possessing formal reality), it nevertheless requires a cause. 'For if we suppose that an idea contains something which was not in its cause, it must have got this from nothing.'[29] But something cannot come from nothing.[30] So, an objective being, though not an extra-mental being, nevertheless will require a cause.

Descartes's example of the idea of the intricate machine in his reply to Caterus is helpful. There is no question about the cause of an idea if what we mean by *idea* here is that it is a mode of a mind. The mind is the idea's cause insofar as the mind (as substance) is the cause of its own operations. And, as he says (in a passage quoted earlier), regarded as such, there are no recognizable differences between his ideas—they all appear to come from within him in the same fashion. Now, what is the cause of an idea's objective being? To be sure, as Descartes admits, the cause could be another idea[31]—or the mind insofar as it is the bearer of the idea. So, the objective being in one idea may be connected to another. Even so, as noted earlier, Descartes says, 'And although one idea may perhaps originate from another, there cannot be an infinite regress here; eventually one must reach a primary idea, the cause of which will be like an archetype which contains formally [and in fact] all the reality [or perfection] which is present only objectively [or representatively] in the idea.'[32] So, he says, in tracking the origin of the objective being in the idea of the intricate machine, eventually one will discover a primary idea (or perhaps some number of them), whose objective being has its origin in some formal being—that is, in some actual or extra-mental object, which, by definition, will possess some level of formal reality. In the reply to Caterus, Descartes discusses two other examples—the idea of God and the idea of a triangle. I will focus on what he says about the idea of the machine, though I will return to the idea of the triangle in a moment.

His discussion of the idea of the intricate machine, as I indicated, suggests that he held there to be a difference between the objective reality of the idea versus its objective being. To account for the *level* of objective reality in an idea we ultimately need only something whose level of formal reality is as great, or greater than, the level of objective reality in question. This seems to be behind the Third Meditation proof for God's existence. But, as also noted, the question of the origin of the level of objective reality in the idea of body yields a different result, for Descartes's mind possesses a level of formal reality great enough to account for the level of objective reality in the idea of body. What the appeal to his mind cannot satisfy is precisely the question over the presence of certain elements

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[29] AT VII, 41; CSM II, 28–9.

[30] Descartes establishes this a bit earlier, at AT VII, 40; CSM II, 28.

[31] Not only does he indicate this in the First Replies passage quoted, but he says as much in the Third Meditation (AT VII, 42; CSM II, 29).

[32] AT VII, 42; CSM II, 29.

(simple natures) presented in his idea of body—extension, shape, size, and so on. Like the objective intricacy of the machine, we must account for the presence of *these* elements—and not just some level of objective reality. Another way to put this is: given an indefinite number of possibilities, why is the idea of *this* object (an extended thing that is shaped, sized, etc.) and no other? What determines the idea's being of the object it is of? Or, in terms of the exchange with Caterus, what determines the presentation of *this particular* objective being? So, in the case of the idea of the sun, what is determining the presence of the *objective sun*, the object presented to me in my idea? Why *this* object? Descartes's answer seems to be that, given that the adventitious idea of the sun is a primary idea, it is the *formal sun*, the sun existing formally in the heavens, that determines the *objective sun*, the sun existing objectively in the mind. No doubt this is what he takes himself to be proving in the Sixth Meditation about the extra-mental existence of corporeal substance: formal body (body existing formally, independent of mind) is responsible for the objective body presented to me in my idea.

Here is the point to stress for our discussion: Descartes seems to hold that the sun existing formally in the heavens exists independent of the mind—in which case, it would seem that if an instance of ideation meets the criterion of (PR), Descartes is a realist with respect to bodies such as the sun. As I mentioned, this is a widely held view, most rigorously articulated in my opinion by Vere Chappell.[33] In Chappell's terms, in cases of what philosophers call veridical perception, the *objective sun* (the sun existing objectively in the mind) has a *counterpart*, namely, the *formal sun* (the sun existing formally in the heavens).[34] In light of our Platonic-Kantian categories developed earlier, Chappell's Descartes is an *empirical realist* with respect to objects such as the sun. Friedman, I think, agrees. The Sixth Meditation argument, as he puts it, establishes an 'objective point of view' with respect to matter.[35] Here, the *objective* aspect of this point of view no doubt is coextensive with empirical reality, or what Moore would emphasize as that which is to be met with in space. The formal sun is an object that can be perceived by multiple perceivers and can exist even though no one perceives it. Given two perceivers who perceive the sun, then, we will count two objective suns—one for each perceiver—but only one formal sun (the sun existing formally in the heavens). So, to reiterate my point, within the *empirical ideality-reality distinction*, and emphasizing the sort of reading forwarded by Chappell, the Sixth Meditation argument establishes that matter is (and, in particular, individual bodies are) empirically real.

Steven Nadler has relatively recently argued for a direct realist reading of Descartes, which challenges the traditional reading. Here, Nadler rejects a *veil of perception* reading,[36] the *veil* referring to the manifold of ideas that is said to stand between extra-mental objects and the mind's eye, so to speak. An idea, in light of a *veil* reading, is sometimes referred to as a *tertium quid*—a third thing. Such a reading says that the 'objects' directly perceived by minds are ideas. Insofar as ideas represent other things (extra-mental objects), the mind only indirectly perceives extra-mental objects. This is a *representationalist* reading, which, as I noted, is the traditional reading of Descartes. The representationalist reading is suggested in Figure 14.1 above (in which the idea is set between the mind and some extra-mental object).

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[33] Chappell (1986). [34] Ibid., p. 193ff. [35] Friedman (1997), p. 158.

[36] Nadler is aligned with Yolton here. John W. Yolton, *Perceptual Acquaintance from Descartes to Reid* (Minneapolis: University of Minnesota Press, 1984). Also, see his 'Representation and Realism: Some Reflections on the Way of Ideas', *Mind* (1987) XCVI (383), pp. 318–30.

Nadler rejects the representationalist reading. According to him, Descartes's view is that the mind directly perceives extra-mental objects such as the sun. Nothing stands between the mind and the extra-mental sun—there is no *tertium quid*. To work this out, Nadler emphasizes a distinction made by Descartes in the 'Preface To The Reader' of the *Meditations*. He, in turn, divines Descartes's view by looking carefully at how two of Descartes's more famous interpreters—Arnauld and Malebranche—understood the distinction, Nadler siding in the end with Arnauld. The distinction offered in the 'Preface' says:

> 'Idea' can be taken materially, as an operation of the intellect, in which case it cannot be said to be more perfect than me. Alternatively, it can be taken objectively, as the thing represented by that operation; and this thing, even if it is not regarded as existing outside the intellect, can still, in virtue of its essence, be more perfect than myself.[37]

Here, Descartes says that there are two ways of regarding or conceiving an idea. The first way of regarding an idea is as an *act* of ideation. This is to take the idea *materially*. The second way of regarding an idea is as representational content—something that is brought about by way of this act. This is to take the idea *objectively*. In terms introduced earlier, it appears that Descartes is saying that the objective being of an idea is related in an important way to a mind's act of ideation—the act, we might say, plays an important role in *producing* the objective being.

According to Nadler, all ideas are, properly speaking, acts of the mind. On his view, the ideational act, or act-idea as he calls it, is the sole bearer of the representational content.[38] 'Act-ideas have a content,' he says, 'but they are not directed towards this content.'[39] In all cases of direct perception act-ideas are directed towards extra-mental objects. The content of an idea—the idea taken objectively—is how the ideational act is experienced from the mind's point of view. This is so because all representational content is *in* the act—it is, if I am reading Nadler correctly, a kind of act-generated 'feedback' experienced by the mind in its directly perceiving the extra-mental object.[40] The mind is not directed at this content (the mind is instead directed at the extra-mental sun, for example), but the content is instead directed at the mind. A representationalist reading, by contrast, claims that the mind is essentially directed at the representational content, where the idea (understood as representational content) presumably would be directed at the extra-mental object (this might be one sense in which the idea represents *that* object). Now, if this 'feedback', or representational content, is what the mind immediately perceives (it is what immediately 'hits' the mind, so to speak, as a consequence of its being directed at some extra-mental object), how is Nadler's reading different from a representationalist reading? Without some qualification, it would appear that in the case of the idea of the sun, for instance, the idea—that is, the idea taken objectively—will stand between the mind's eye and the extra-mental sun.

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[37] AT VII 8; CSM II 7. [38] Nadler (1989), p. 146. [39] Ibid., p. 168.

[40] Chappell says something similar, holding that all ideas (taken materially) represent ideas (taken objectively). The idea (taken objectively), he says, is the *content* of the idea (taken materially). Chappell (1986), p. 190.

Though published several years before Nadler's study of Descartes, George Pitcher has offered an analysis that is of some help in coming to understand Nadler's view. Let me briefly introduce it here. In his examination of Berkeley, Pitcher tries his hand at interpreting Berkeley as holding what might be called an adverbial theory of ideation.[41] I have argued elsewhere for reading Descartes this way with respect to the sensible qualities.[42] Pitcher's analysis begins by considering the following sorts of claim:

1) Jones kicks a tree.
2) Jones turns a somersault.

On their surface, both claims appear to have identical grammatical (or logical) structures: two things that are related. But they do not have identical structures. In the first claim, Jones and a tree are the two things, and they are related by the action of kicking. Here, Jones is said to be doing the kicking, where the tree is that on which Jones acts. What is more, this particular tree, the one that Jones kicks, is taken as something that can (and presumably does) exist independently of Jones. The second claim's structure is importantly different. If we think about it, there really is only one thing in play, namely, Jones. The somersault is not a 'thing', at least not in the sense that Jones is a thing, and it is not something on which Jones is acting. What is more, unlike the tree in case (1), the somersault, in case (2), is not independent of Jones. Rather, turning the somersault, says Pitcher, is *a way of being* Jones. That is, turning a somersault here is identical with being Jones (at least *while* he turns the somersault). As I said, Nadler's view is helped by Pitcher's analysis, which tells us that 'mind ***m*** perceives body ***b***' should be read along the lines of (2) above. Thus, ***m***'s perceiving ***b*** is simply a way of being ***m***, in the way that Jones's turning a somersault is simply a way of being Jones. Understood this way, there is no *third thing*—an idea—standing between mind *m* and body *b*. Rather, the idea is simply a way of being mind *m*. And, if we read Descartes this way, Nadler insists, Descartes is not a representationalist. To the extent that Descartes holds that body *b* is directly perceived by mind *m*, where nothing stands between them, Descartes looks to be a direct realist.

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[41] Pitcher deals specifically with this in Chapter XI, 'Berkeley's View of Mind', in his book, *Berkeley* (New York: Routledge & Kegan Paul, 1977). Also see his 'Berkeley on the Mind's Activity', *American Philosophical Quarterly* 18 (1981), pp. 221–7.

[42] Smith (2005).

Nadler reads *formal* and *material* as being synonymous,[43] and subsequently reads the Third Meditation's formal-objective reality distinction and the Preface's material-objective distinction as being one and the same. Nadler rejects any essential connection between the idea's representational content and the extra-mental object at which the mind is directed.[44] This, I believe, spells trouble for his view. For, as he sees it, even though the act is directed at an extra-mental object, the extra-mental object plays absolutely no role in determining the representational content. By his lights, an idea's representational content is determined solely by the ideational act itself. This, I think, severs the tie established in the formal-objective reality distinction, between an idea's objective reality and the formal reality of an extra-mental object. Recall, this essential connection is established by way of Track2. And, (PR) is based on this. What Nadler fails to see, I think, is that within the contexts of the distinctions, *formal* and *material* are not synonymous. The distinctions are emphasizing importantly different relations. I will explain.

When applied to ideas, the formal-objective reality distinction emphasizes two relations: the relation between an idea *as* mode and its substance (a mind), and the relation between an idea *as* representational content and the formal reality of some object (which, when not one's own mind, is an extra-mental object). The first relation is emphasized in our taking the idea formally, the second relation is emphasized in our taking the idea objectively. The material-objective distinction is importantly different. It emphasizes only one relation—a relation *not* emphasized in the formal-objective reality distinction. The material-objective distinction emphasizes the relation between an idea *as* act and this very same idea *as* representational content. The distinction is telling us that the idea *as act* plays an important role in producing the idea's representational content. What Nadler does, I think, is wrongly conclude from this that the idea *as act* is playing the *only* role in producing the idea's representational content. Consider again the idea of the sun. No doubt, insofar as the objective sun is a mental being, there is an important sense in which it depends on the mind. The material-objective distinction, I contend, tells us the sense in which the objective sun depends on the mind. By contrast, the formal-objective reality distinction tells us that the objective being, in this case the objective sun, is determined by the formal reality of the sun—it is determined by the formal sun.

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[43] Nadler (1989). Marjorie Grene also reads it this way. Marjorie Grene, *Descartes* (Indianapolis: Hackett Publishing, 1998), p. 177. This is a reprint of the 1986 University of Minnesota book.

[44] Nadler (1989), p. 169.

Paul Hoffman has also developed a direct realist reading of Descartes. He specifically contrasts it to Nadler's.[45] Hoffman takes issue with Nadler's claim that act-ideas are not directed at representational content. According to Hoffman, act-ideas *are* directed at representational content—they are directed at the idea taken objectively. A challenge that immediately arises, of course, is how to read Hoffman's view as a direct realist reading, since in the case of the idea of the sun, the idea (taken objectively) looks to stand between the mind and the extra-mental sun. His answer to this challenge is rooted in his contention that it is such objects as the sun, for instance, that possess the two kinds of reality mentioned earlier—formal and objective reality. Objective reality is not 'attached' to minds, but to things such as the sun.

Insofar as the sun exists independent of the mind, it possesses formal reality. So far, so good. About where to locate ontologically the objective reality of the idea (or, in this case, the objective sun), Hoffman puts a great deal of weight on a remark that Descartes makes to Caterus. Descartes says that by the phrase 'idea of the sun,' he means 'the sun itself existing in the intellect—not of course formally, as it does in the heavens, but objectively existing, i.e., in the way in which objects normally are in the intellect.'[46] This, according to Hoffman, suggests that it is the sun that also possesses objective reality. For, the objective sun, as we called it earlier, even though it is the *idea* of the sun (the idea taken objectively) is nevertheless the sun itself. 'The sun as it exists objectively,' Hoffman says, 'is able to *represent* the sun as it exists formally in the heavens precisely because it is the same thing that has these two different modes of existence.'[47] Hoffman's direct realist reading, then, seems to go as follows. When perceiving the sun, one is directly aware of the objective sun. But, since the objective sun and the formal sun (or, the sun's objective reality and formal reality) are simply two different aspects of the *same* object—the sun—when directly perceiving the *objective sun*, one in essence directly perceives *the sun*. Understood this way, Hoffman takes Descartes to be a direct realist. And, given that objects, such as the sun, can exist independent of any particular mind, and can be perceived by multiple perceivers, he reads Descartes as an empirical realist with respect to individual bodies.

One problem with Hoffman's view, I think, stems from his attaching objective reality to the sun. On the traditional reading of Descartes, objective reality is ontologically located in the mind—though in cases of primary ideas this kind of reality can be traced (via Track2) to the formal reality of extra-mental objects. Even Nadler's view, which also challenges the traditional reading, nevertheless agrees with the traditional reading when it comes to where to locate objective reality ontologically—he locates it in the mind. Hoffman suggests that his view has its roots in Aquinas. And, certainly, the view even looks to be friendly to the one developed by Malebranche—this is so, I think, insofar as Hoffman must hold that ideas are not in finite minds but are instead the 'objects' at which such minds are directed. The problem, as I see it, is this. Let the sun be that to which we attribute formal reality. This, as we called it earlier, is the formal sun. This being can (and presumably does) exist independent of any finite mind. Let's follow Hoffman and attribute objective reality to the sun—the *same* object to which we attributed formal reality. Possessing objective reality, Hoffman's sun is now the objective sun. The formal and objective 'suns' are simply two aspects of the same object—the sun. Now, given that the sun can (and presumably does) exist independent of any finite mind (it is the formal sun after all), and objective reality is attributed to it, then the objective sun can (and presumably does) exist independent of any finite mind. So, if the objective sun is the *idea* of the sun taken objectively, then there is a sense in which the *idea* of the sun can exist independent of any finite mind. As I said, Malebranche may be able to accommodate Hoffman here, but I do not believe that Descartes can.

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[45] Paul Hoffman, 'Direct Realism and the Objective Being of Ideas', *Pacific Philosophical Quarterly* 83 (2002), pp. 163–79.

[46] AT VII, 103; CSM II, 75.

[47] Hoffman (2002), p. 168, italics mine.

Consider an analogy that was discussed in Part I: the analogy of Socrates standing before a mirror. Both Socrates and the mirror are necessary for the image of Socrates (the image located on the mirror's surface). For, if we destroyed either Socrates or the mirror, we would also destroy the image of Socrates. Even so, we say that the image is 'of' Socrates, and not 'of' the mirror, if what we are pointing out is *what* this image represents. The image of Socrates depends for its being the thing it is (a representation *of Socrates*) on the being of *Socrates* in a way that the image does not depend for its being the thing it is (a representation *of Socrates*) on the being of the mirror. Of course, since the image is located on the surface of the mirror, there is something about the image that does depend on the mirror. So, if we wanted to point out the 'owner', so to speak, of the image, we might say that the image is 'of' the mirror (in the way that we might say that the surface is of, or belongs to, the mirror).

The analogy is helpful in coming to see where both Nadler and Hoffman appear to go wrong. How the analogy of the mirror aligns with Descartes's theory of ideas, at least as traditionally understood, should be obvious. The image on the mirror's surface is analogue to an idea. The mirror itself, the bearer of the image, is analogue to the mind (which is bearer of an idea). Socrates, standing before the mirror, is analogue to an extra-mental object—a formal being that exists independent of the mind. We might say that the shiny surface (the mirror's surface) is the *medium* of representation, or, *that by way of which* Socrates is represented by the mirror. Accordingly, then, we might cast the shiny surface as analogue to an idea taken *materially*—as the *act* of representation. It is analogous to that by way of which objects are represented by the mind.

Here is how the formal-objective reality distinction might be applied. Begin with the image of Socrates on the mirror's surface. It is analogue, recall, to the idea of Socrates. Taken *formally*, we say that the idea is a mode of the mind. By analogy, we should say that taken *formally*, the image is a mode of the mirror. Taken objectively, where we focus on *what* the image is an image *of*, we say that the idea's objective reality (its origin) is traceable to the formal reality of Socrates. In terms of *objective being*, we might say that objective Socrates is traceable to formal Socrates, so that were there no formal Socrates, there could be no objective Socrates. According to (PR), the idea represents Socrates precisely because the idea's objective reality has its origin in the formal reality of Socrates. By analogy, the mirror image represents Socrates because the image's objective reality has its origin in the formal reality of Socrates. Were Socrates not standing in front of the mirror, there could be no mirror image of Socrates.

In light of this analogy, here is where I find problems for Nadler and Hoffman. According to Nadler, recall, there is absolutely no connection between the representational content of an idea and the extra-mental object. The ideational act itself is solely responsible for determining the representational content. In light of the analogy, this amounts to claiming that the shiny surface of the mirror is solely responsible for determining what the mirror image represents. Socrates, on Nadler's view, plays absolutely no role in determining that the mirror image represents Socrates. So, there could be a mirror image of Socrates even though Socrates was not standing before the mirror. If this is where Nadler's view leads, it is surely wrong. Hoffman fares no better. Recall that he attaches or attributes the idea (taken objectively) to such extra-mental objects as the sun. In light of the analogy, this amounts to claiming that the mirror image is to be located in (on?) Socrates. That is, Hoffman's view seems to allow there to be mirror images of Socrates independently of there being any mirrors. The troubles sketched here are connected to Nadler's and to Hoffman's reading Descartes as a *direct realist*. The criticism does not challenge the view that Descartes was an empirical realist when it came to individual bodies. Descartes's empirical realist position seems to stand regardless of whether he held the direct realist or the representationalist view of ideas.

There is no doubt that Descartes held *some* form of realism with respect to matter, though as my assessment above suggests, it is not clear that it was a form of realism tied to direct realism. To go on the record, I agree with the traditional reading, which casts Descartes as a representationalist. Be that as it may, as I just stated, one sense that seems to hold (for either reading) is that for Descartes, body (at least within the context of the Sixth Meditation proof) is empirically real. Recent commentators have begun to argue that Descartes was not a realist about *individuated* or *individual bodies*. Rather, he was an idealist with respect to them. They argue that he would not have taken the sun to be an object capable of existing independent of thought. Although these commentators take themselves to be arguing against the 'many bodies' view, a view embraced by Chappell, Nadler, and Hoffman, I think that in their emphasizing the *conditions for the conceivability* of individual bodies, their arguments are actually tracking a different strain of argument in Descartes. I do not think that they are *opposed* to the empirical realist reading of Descartes, and they are mistaken if they think that their views are opposed to it. In my view, they have discovered a transcendental reading of Cartesian matter. I will say more about this in the following chapters. For the moment, I want to look at how this 'one body' view is argued for by its proponents, and sketch out some problems for the view that will help build a case against reading the view as a competing empirical realist reading of Descartes.

Lawrence Nolan argues for the 'one body' view. Based upon Descartes's insistence that the essence of body is extension, Nolan argues:

> Contrary to the traditional Aristotelian conception of the universe, this means that there are no natural kinds: a tiger, for example, does not have a distinct essence from, say, a dandelion. Similarly, a triangular-shaped body does not have an essence that is distinct in kind from that of a square-shaped body. They are all purely extended things whose differences are to be cashed out mechanically (i.e., in terms of differences in size, shape, position, etc.) and not essentially.[48]

Given that there are no 'things' such as suns (because there are no natural kinds), it follows trivially that there will be no suns in extra-mental reality. If we accept that there is no such thing as the sun, we nevertheless might maintain that there are individual bodies. Fine. But how are they individuated? Not by essence, Nolan argues, since they all have the same essence—extension. By shape then? But if what distinguishes them from one another is shape, we have drawn only a *modal* distinction between them.[49] They do not appear to be *really distinct* from one another, which is a requirement for finite substancehood on Descartes's view.[50]

Thomas Lennon approaches the view from a slightly different angle. He has recently argued that motion is ideal for Descartes, and since it does not exist independent of mind, this means that motion is not *real*.[51] And, if motion is not, and motion is what individuates bodies in the plenum, then individuated bodies (such as the sun) are not real.[52] They are ideal. He concludes:

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[48] Lawrence Nolan, 'The Role of the Imagination in Rationalist Philosophies of Mathematics', in *A Companion To Rationalism*, edited by Alan Nelson (Oxford: Blackwell, 2005), p, 235.

[49] AT VIIIA, 29–30; CSM I, 213–14.

[50] AT VIIIA, 28–9; CSM I, 213. There is a difficulty with Descartes's example of the division of a body into its parts, where he does more than to suggest that the parts are really distinct from one another. This would suggest that they are individual corporeal substances. Any interpretation of this passage, however, must take into serious consideration what Descartes means by the phrase 'as delimited by us in our thought', which he uses to characterize how the parts get divided (or individuated). For an account that makes sense of this passage, see Kurt Smith and Alan Nelson, 'Divisibility and Cartesian Extension', in *Oxford Studies in Early Modern Philosophy*, vol. v, edited by Daniel Garber and Steven Nadler (2010), pp. 1–24—an account that argues that the distinction drawn between the parts is modal.

[51] Lennon (2007), p. 30ff. Also see his 'Descartes' Idealism', in *Philosophie et Culture*, Actes du XVIIe Congrès Mondial de Philosophie (Montreal: Editions de Beffroi, Editions Montmorency, 1986) vol. IV, pp. 53–6; Alice Sowaal, 'Cartesian Bodies,' *Canadian Journal of Philosophy* (2004), vol. 34, no. 2, pp. 217–40. Also see her 'Idealism and Cartesian Motion', in *A Companion to Rationalism,* edited by Alan Nelson (Oxford: Blackwell Publishers, 2005), pp. 250–61.

> There may be *body* that is independent of mind, but there are no *bodies* that are independent of mind. Or, as Descartes might also put it, without mind there can be *res extensa* (the one extended thing) or *extensio* (extension), but there cannot be *res extensae* or *extensa* (extended things). In my view, Descartes is not only committed to this, but actually holds it.[53]

If Lennon is right, Descartes will not be a realist with respect to individuated bodies such as the sun. He will be an idealist. Even so, he may still be a realist with respect to *res extensa*. But, what kind of realist? An empirical realist? What would that even mean?

Like Nadler, Nolan and Lennon put a great deal of weight on an idea being understood as an *act* of ideating. The idea of the sun, for example, is, on Nolan's reading of Descartes, *a way of regarding* extension. This holds for mathematical ideas as well. The idea of a triangle will be a way of regarding extension.[54] So, the idea of the sun or the idea of a triangle is an act, a way of regarding, and is directed at *res extensa*. Extension, or *res extensa*, it seems, is not an idea, but is the object (mind-independent object) at which the idea *as act* is directed. Here, Nolan does more than to suggest that there are two 'levels' of object (of ideation), one *overlying* and the other *underlying*. In the example of the idea of the sun, the sun is the overlying object, and *res extensa* the underlying object. In the example of the idea of the triangle, the triangle is the overlying object, and *res extensa* the underlying object. Lennon (and as we shall see shortly, Alan Nelson) seems to also hold this.[55] But, the view seems to suggest that the overlying object, in being immediately perceived, stands between the mind and the underlying object. To avoid this sort of representationalist reading, Nolan (and I think Lennon and Nelson) casts the underlying object as that which the idea is really 'of'.[56] That is, after philosophical reflection, if we asked what our idea of the sun is an idea *of*, the right answer would be that it is really of *res extensa*. This also holds for the idea of the triangle.

In both cases, Nolan says, the overlying object is simply *a way of regarding* the underlying object. The overlying object, in other words, is that *by way of which* we perceive what I am calling the underlying object. It is the medium or operation of perception (saying it this way, we take the idea materially). But it is not the *object* of perception. The object of perception is *res extensa*. Adopting adverbial terms, Nolan says that the idea of a triangle, for instance, is our regarding *res extensa* triangularly—the point being that the object being perceived (or so regarded) is *res extensa*. It is important to keep in mind that for Nolan (and Lennon and Nelson) the overlying object is not something that lay literally on top of the underlying object (though we could take it this way). Rather, the overlying object is the metaphysical vehicle *by way of which* the underlying object is represented by the mind. Without qualification, this view suggests that we directly perceive *res extensa*. And so, *prima facie*, it would appear that Nolan (though perhaps neither Lennon nor Nelson) is offering a direct realist reading of Descartes.[57]

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[52] Lennon (2007). [53] Ibid., p. 30.

[54] Nolan (2005), p. 240. Also see Nolan, 'Reductionism and Nominalism in Descartes's Theory of Attributes', *Topoi*, vol. 10, no. 2 (September 1997), pp. 129–40; and 'The Ontological Status of Cartesian Natures', *Pacific Philosophical Quarterly*, vol. 78, no. 2 (June 1997), pp. 169–94.

[55] See Lennon (1986), (2007). [56] Nolan (2005), pp. 234, 235.

This would be a strange sort of direct realism indeed. The 'object' directly perceived would be *res extensa*, though strangely enough *res extensa* is, on this view, never presented *to* the mind. Rather, what we are presented with are suns and triangles, and so on. Nolan tries to alleviate this problem, as we saw, by casting the 'presented' things as *ways of regarding*—so, they are not images standing between the mind and *res extensa*, but are simply *ways of being the mind* in its activity of perceiving *res extensa*. This looks a bit like Nadler's view, I think. Even so, at least Nadler had available a metaphysical ground for our having, say, an idea of the sun—the ground was the formal sun, the sun existing formally in the heavens, independent of the mind (though, as noted, he did not take advantage of this—so his view would suffer from the same problem). Chappell's view (which aligns with the traditional reading) avoids the problem altogether. Nolan (and Lennon and Nelson) does not have this. So, what accounts for my regarding extension *sun-ly*? That is, why do I *ever* have the idea of the sun? (I am thinking specifically of the adventitious idea of the sun.) What accounts for my having the (adventitious) idea of the moon? Why the sun one moment, the moon the next? Is it simply due to *my* mind? If it is, it certainly is not something of which I am aware. As Descartes is clear to point out about his adventitious ideas of bodies (such as the sun and moon), 'these ideas do not depend on my will', and as a consequence, 'they do not depend simply on me'.[58] These ideas strike him even *against* his will, for 'Frequently I notice them even when I do not want to.'[59] If *res extensa* is that 'unbounded monolith', as Grosholz calls it, what accounts for *my* (secretly?) regarding it now as the sun, now as the moon, now as the fire (next to which I am sitting)? For, on Nolan's (and Lennon's and Nelson's) view there appears to be nothing in the unbounded monolith *but* extension in length, breadth, and depth.

Reconsider the idea of a triangle. We know that Nolan wants to recast the idea of the triangle as a way of regarding extension. To be sure, Descartes says that, unlike the adventitious ideas of the sun and moon, the will can directly summon the idea of the triangle (or any geometrical figure). So, if I am having the idea of the triangle, I can account for its presence by an appeal to my will—I wanted to think of the triangle. Even so, as Nolan has been clear to establish, Descartes will not *ground* the true and immutable nature of the triangle in any finite mind—the triangle has a true and immutable nature that transcends any and all finite minds. So, even though I can think of the triangle at will, my will is not responsible for the triangle's nature. What is responsible? Nolan argues that it is extension. Extension is what grounds the triangle's nature. However, by this Nolan seems to mean that extension is a necessary condition for the *intelligibility* of the triangle (and all other geometrical figures). The trouble is that on Nolan's view triangles are nowhere to be found in *res extensa*. So, if the true and immutable nature is not formally in a finite mind, and it is not formally in extension, where is it (ontologically speaking)?

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[57] Lennon's view, it seems to me, pushes for the more extreme idealist reading.

[58] AT VII, 38; CSM II, 26.

[59] Ibid.

To Caterus, recall, Descartes introduced his example of the idea of an intricate machine. I mentioned that in connection to this he also says something about the idea of the triangle. Here is the connected remark:

> But my shrewd critic sees all this quite well, and he therefore concedes that we can ask why a given idea contains such and such objective reality. His answer is that, in the case of all ideas, what I wrote in connection with the idea of a triangle holds, namely that 'even if perhaps a triangle does not exist anywhere, it still has a determinate nature or essence or form which is immutable and eternal.' And this, he says, does not require a cause. But he is well aware that this is not an adequate reply; for even if the nature of the triangle is immutable and eternal, it is still no less appropriate to ask why there is an idea of it within us.[60]

Here, Descartes appears to think that there is something other than his act of *willing* that accounts for the idea's being of *a triangle* and not of some other thing. To be sure, if I want to think of a triangle, and do, it seems perfectly acceptable to answer the question 'why is the object you are thinking of, *a triangle*?' by an appeal to my will. I am thinking of a triangle because I *wanted* to think of such a thing. But the reply to Caterus suggests that this is not satisfactory if what we want to know is why this object is *a triangle* and not some other object. In terms established earlier, Descartes is asking: what accounts for the objective triangle? What determines the idea's being of *this* object?

This, I think, is similar to Descartes's concern over the objective intricacy in the idea of the machine. What accounts for it? His answer, at least within the confines of establishing that body is empirically real, is that a formal intricacy (ontologically located outside the mind) accounts for the objective intricacy. This holds for the idea of the sun as well. What accounts for the objective sun? His answer: the formal sun—something that exists independent of the mind. The problem for Nolan, if we insist on reading him as advocating for an empirical realist reading of body (the one body, *res extensa*), while advocating for an idealist reading with respect to individuated bodies, is that he has lost any account of the origin of objective beings—such as the sun and the triangle. How is it that our ideas present *these* objects to the mind? His only remedy seems to be that the objective beings are what they are because one's mind thinks them (which does not seem to be a remedy at all), or that the objective beings, though not in *res extensa*, are in the mind of God. As even Kant saw it, 'outer objects are quite indispensible' when trying to account for the determination of thought—where the account will be lacking if all one makes available in the account is mind.[61] Here, like Hoffman's view, Nolan's would seem to better align with Malebranche than with Descartes. The way to read Nolan (and Lennon and Nelson), I think, is not as offering an empirical realist view of body (in competition with Chappell, Nadler, and Hoffman). By abandoning our trying to make Nolan's view an empirical realist reading of Descartes, we can avoid the problems mentioned above. Though to do this, we will have to rework some of the view. Let me explain.

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[60] AT VII, 104; CSM II, 76.

Nolan's view, it seems to me, ultimately has its roots in Descartes's theory of *abstraction*. As noted earlier, focusing on an idea's element while *ignoring* the others is what Descartes calls abstraction.[62] Although we can abstract number, say, from extension, we cannot completely separate or *exclude* it from extension without rendering number or quantity unintelligible.[63] Exclusion of extension from number would require us to conceive both, and then to clearly and distinctly conceive each on its own. The trouble will be in conceiving number without *also* conceiving extension. This holds for geometrical figures as well. We can ignore extension while focusing on shape, which is what we do when abstracting, but we cannot exclude extension without rendering geometrical figures unintelligible. To be able to exclude an element from others is to show that this element is *really distinct* from the others—it can exist independently of the others.[64] It should not go unnoticed that what holds for numbers and geometrical figures holds equally for such things as the sun. We can certainly ignore extension when perceiving or thinking about the perceived sun (by focusing on its shape or position, for instance), but we cannot completely exclude extension from the idea of the sun. To do so would render the sun (understood as a body) unintelligible. The underlying object, extension, is a necessary condition for the intelligibility of the overlying object, the sun (in the other examples, a number or a geometrical figure). In emphasizing extension's being a necessary condition for the intelligibility of not only mathematical objects (such as numbers and geometrical figures), but for individual bodies of empirical reality (such as the sun), Nolan (Lennon, and I think Nelson) appears to be on to a line of reasoning in Descartes that looks suspiciously transcendental in scope. I will now turn to developing this.

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[61] Kant, *Critique of Pure Reason*, B 277 (Norman Kemp Smith, p. 246).

[62] Letter to Mesland, 2 May 1644, AT IV 120; CSMK III 236. For excellent discussions of this, see Nolan (June 1997); (September 1997).

[63] AT VIIIA, 44–5; CSM I, 226.

[64] AT VIIIA, 28–9; CSM I, 213.

15

Empirical Reality and Intelligible Matter

Friedman says that once Descartes establishes the 'objective point of view' which, recall, is coextensive with the concept of empirical reality, the Sixth Meditation inquiry into matter takes an interesting turn. Friedman says:

> Once we have attained this purely objective point of view, however, we find that there is a second, and apparently entirely independent, way of considering the all-important transition from the ideal or Platonic existence of Cartesian geometrical extension to its real or actual existence. This is the point of view of the all-perfect and all-powerful God, who must create and conserve geometrical extension in existence by 'the same power and action,' just as he creates and conserves me in existence as a finite thinking substance. Whereas, from the point of view of the sixth *Meditation*, really existent Cartesian matter appears as the cause of my subjective sensations, this same really existent matter, from a purely objective point of view, now appears as the effect of God's all-powerful creative will. From my subjective point of view, the difference between ideal or Platonic existence and real existence in nature is characterized in terms of a causal influence on my sensations. From a purely objective point of view, however, the transition between ideal or Platonic existence and real existence in nature is rather affected by God himself. These two ways of considering the transition from the ideal existence of matter to its real existence reflect the unique role of Cartesian matter as a necessary intermediary between my sensory ideas and the perfection of God.[1]

Here, Friedman suggests a second and equally important way of reading what is being established by Descartes in establishing the existence of matter. Notice that matter is cast as *necessary*—it stands metaphysically between finite mind and God. In Part III, we explored one sense in which matter serves this metaphysical function. We learned that *if matter exists, mathematics is intelligible*. Matter was there shown to be sufficient for guaranteeing the intelligibility of mathematics. What we are exploring now is Friedman's insight into the existence of matter in light of God's *will*. What we will learn is that the existence of matter is a necessary condition for the intelligibility of mathematics. This, it seems to me, points us in a direction that takes us away from the Visible or Sensible section of the divided line, towards the Intelligible section. In Kant's terms, Friedman's insight suggests a way of reading Descartes that shows matter to be transcendentally ideal or real, depending on how we interpret certain texts.

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[1] Friedman (1997) pp. 158–9.

Friedman does not say much more about this particular insight than what we find in the above passage. But there is a remedy. Several others have also had the insight that Friedman expresses. I will look briefly at what they have said, afterwards returning to Friedman.

Alan Nelson's interpretation of the Cartesian view, mentioned a number of times above, is rooted in an examination of the exchange between Arnauld and Leibniz.[2] On this account, Descartes's adherence to the doctrine of divine simplicity appears to commit him conceptually to a version of what Nelson calls modal actualism—the view that 'all philosophically interesting uses of possibles [are] analyzable into facts about actually existing things'.[3] The doctrine of divine simplicity tells us that God is a simple being. To be sure, we can conceive of God's having an intellect, say, along with his having a will, which would appear to show that he is complex after all. But, our conceiving such distinctions is the result of *our* finite intellect. For example, regarding Lincoln as sixteenth US President and, say, as husband of Mary Todd, does not show that there are *two* Lincolns. The distinction here is produced by us, in our conceiving Lincoln one way and then another.[4] Nelson notes that, according to the doctrine, the exercising of God's will is also coextensive with the creative act.[5] This sets up the following form of argument: assume that *A* is a finite being. If God conceives *A*, God wills *A*. And if God wills *A*, God creates *A*. And so, if God conceives *A*, God creates *A*.[6] The significance of this line of reasoning to our discussion should be obvious. For, God's conceiving extension entails God's creating it. And so, it would not be possible for God to conceive extension (thus making mathematics intelligible) and yet extension does not exist. And, as such, extended substance would still be said to be only *possible*, since it depends ontologically on God for its existence. This, even though it *actually* exists. In Cartesian terms, the claim that the idea of extension contains only possible existence is not supposed to imply that it does not actually exist. *Possible existence* simply indicates extension's ontological dependence on the divine being.

In a related paper, Nelson and David Cunning fill in the view. Along the lines of Nelson's earlier article, *possible* or *contingent existence* is cast as an ontological relation of dependence—finite substances are *possible* in the sense that they depend for their existence on the existence of God.[7] Again, it is important to understand that by *possibles* here, Nelson and Cunning are talking about actual existents. They are actual existents that depend for their being on the being of God. It is this dependence relation that 'possible' is picking out. And so, according to this view, it will be misleading to talk about *possibly existing* as opposed to *actually existing*. Anticipating the objection that there is an important difference (for Descartes) between these two phrases, they argue:

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[2] Alan Nelson, 'Cartesian Actualism in the Leibniz-Arnauld Correspondence', *Canadian Journal of Philosophy*, vol. 23, no. 4, (December 1993), pp. 675–94.

[3] Ibid., p. 676.

[4] Some might insist that, technically speaking, this is a modal distinction. But my point is that conceiving *x* as *f* and as *g* in no way guarantees that there are two *x*'s.

[5] Nelson (1993), p. 686. Nelson relies on a 2 May 1644 letter to Mesland and a passage from the *Principles*.

[6] Ibid., p. 687.

[7] David Cunning and Alan Nelson, 'Cognition and Modality in Descartes', *Acta Philosophica Fennica*, 64 (1999), pp. 137–53. They make the claim on p. 143. In the earlier paper, Nelson (1993) makes it on p. 684.

> This objection presupposes that a particular perception of the merely possible existence of a thing differs intrinsically and phenomenologically from a perception of the actual existence of the same thing. This is extremely implausible. Suppose that one is clearly and distinctly perceiving extension. The idea of extension, of course, contains contingent existence. Now suppose that one is perceiving extension as actually existing, perhaps after reviewing a proof to that effect. What is the difference? In the first case, we surely do not clearly and distinctly perceive extension accompanied by a spooky halo signifying *merely* possible existence. Another way must be found for explaining the difference between the meditator who, on the one hand, knows that extension actually exists and the meditator who, on the other hand, is still hyperbolically doubting it. There is no phenomenological difference between perceiving actual existence and perceiving some supposed, merely possible existence. There is only, we reiterate, the fact that perception of dependent existence is included in the clear and distinct perception of a thing.[8]

On this view, one is simply confused if one *doubts* the actual existence of extension. When clear and distinct, one conceives that this actual existent is a contingent existent in the sense that its existence depends utterly on God. But for all that, when clear and distinct, one conceives that extension actually exists—it is a mind-independent substance. What is the textual and philosophical support for this reading of Descartes? Let's look.

On the Nelson–Cunning view, the salient formulation of Descartes's metaphysical position tells us that the existence of body is only *conceptually* or *rationally distinct* from its extension. This allows Descartes to hold that even in cases where a finite mind clearly and distinctly perceives (or conceives) extension, it will be impossible to perceive (or conceive) it while excluding from the occurring idea that extension (or the extended substance) endures or exists. Descartes says:

> Such a distinction is recognized by our inability to form a clear and distinct idea of the substance if we exclude from it the attribute in question, or, alternatively, by our inability to perceive clearly the idea of one of the two attributes if we separate it from the other. For example, since a substance cannot cease to endure without also ceasing to be, the distinction between the substance and its duration is merely a conceptual one.[9]

Given that an attribute is only conceptually distinct from substance, it follows that we will not be able to conceive corporeal substance in exclusion of extension (its principal attribute). And, given that attributes (of the same substance) are only conceptually distinct from one another, it follows that when we conceive of extension, we conceive of an *enduring* corporeal being—a substance. In other words, when clearly and distinctly conceiving body, we *ipso facto* conceive of it *as* existing. So, the very conception of matter forces on us the claim that it is real.[10]

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[8] Cunning and Nelson (1999).

[9] AT VIIIA, 30; CSM I, 214.

Now, what is at the bottom of *our* inability to clearly conceive (or perceive) certain attributes as being really distinct from one another? Why is it that we cannot conceive of extension, for instance, *and* at the same time clearly and distinctly conceive (or at least clearly entertain) that it does not exist? The answer, according to Nelson and Cunning is: God is at the bottom of all this. For, it is God who guarantees the existence of extension by his conceiving it, which explains why it is that *we* cannot conceive of extension as being independent (or really distinct) from its existence or duration. *We* cannot conceive of a real distinction between these attributes because when God conceives extension, his doing so is coextensive with his creating it. Descartes says something similar to Mersenne,[11] accounting for the 'necessity' of mathematical truths, for example, by an appeal to God's will. It is God's willing that all radii of a circle are equal, for instance, that accounts for the truth's 'necessity'. Because God wills this mathematical truth, *we* cannot conceive its opposite or 'undo' it by thinking that at least one of the radii of a circle is unequal to the others. Similarly, God's will establishes the internal relation between a substance and its attributes, and the relation between the attributes. So, the internal metaphysical connection between extension and duration, for instance, cannot be broken or undone, since God is that which makes the connection. Like I said, this is a sense in which matter (extension) is taken to be real. But, it does not appear to be the sense in which we took matter to be empirically real.

To work out what I think this new sense is (which I have suggested is a transcendental sense), let's begin with what Descartes says we can clearly and distinctly perceive. In *Principles*, Part I, Article 54, Descartes says:

> Thus we can easily have two clear and distinct notions or ideas, one of created thinking substance, and the other of corporeal substance, provided we are careful to distinguish all the attributes of thought from the attributes of extension. We can also have a clear and distinct idea of uncreated and independent thinking substance, that is of God.[12]

There are at least three clear and distinct ideas: the idea of finite mind, the idea of (indefinite) body,[13] and the idea of infinite mind. Finite mind and indefinite body are both understood to be created, whereas the infinite mind is understood to be uncreated. Descartes will never characterize the idea of the sun, for example (or the ideas of other 'natural' objects), as being clear and distinct. When the sun is understood *as a body*, the nature of which is extension, the idea is as clear and distinct as it is going to get—but it is important to keep in mind that the clear and distinct idea of the sun *as body* is not anything like the obscure and confused idea of the sun that presents the sun as yellow, hot, etc. (Nolan's argument against Descartes's holding natural kinds is relevant here.)

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[10] Kant makes a similar claim in his *Critique of Pure Reason*, A 375-A 381 (Norman Kemp Smith, pp. 349–52).

[11] AT I, 152; CSMK III, 25.

[12] AT VIIIA, 25–6; CSM I, 211.

[13] Descartes characterizes the limitlessness of extension as indefinite (AT VIIIA, 52; CSM I, 232).

Using Nelson's terminology, to 'clarify' and to 'distinctify' *these* sorts of ideas (adventitious and factitious ideas), we would have to ignore most of what they present, and focus solely on elements that steer us towards some attribute. If we focused on shape, for instance, we would be intellectually steered towards the attribute of extension.[14] This, in essence, would be our coming to see that extension is what *underlies* shape in the sense that it accounts for the *intelligibility* of shape. If we instead focused on other elements, for example colours, our doing so would steer us towards some other attribute, the underlying object of the idea, which according to Nelson would be either mind or union (depending on how we were regarding colour). Again, this would be our coming to see that thought, for example, is what underlies colour in the sense that it accounts for the intelligibility of colour. When engaged in this sort of ideational analysis, notice that the *sun* is analysed into simpler elements—shapes, sizes, colours, etc.—which in the *Rules* were called simple natures. The sun and other such 'objects' simply do not survive the analysis, which reveals another sense, Nelson suggests, in which they are not *real* objects after all. Nelson argues that the true underlying objects, those that *do* survive the analysis, in fact our analysis terminates in them, are either God (infinite mind), finite mind, indefinite body, or even (mind-body) union.[15] These, as he points out, are what Descartes calls the *primitive notions*.[16] These three (and on Nelson's view, four) primitive notions are the true objects of all ideas. They are what our ideas are really *about*. To keep the earlier terminology, the shape is the overlying object, extension the underlying object, where the former is a way of regarding the latter. Notice how this can align with both Nolan's and Lennon's respective views.

So, as we focus on shape, say, extension is brought out as the idea's underlying object; it is what accounts for the intelligibility of shape. This is shown, as Descartes himself suggests, by seeing that in excluding extension from the idea we render shape unintelligible.[17] Depending on how we focus on colour, as I said, either mind or union is brought out as the idea's underlying object. On Nelson's view, the elements of our ideas will be logically rooted in at least one underlying object, of which there are at least three (and on Nelson's view, four)—either

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[14] This component of Nelson's view can be found in his 'Cartesian Innateness', in *A Companion to Descartes*, edited by Janet Broughton and John Carriero (Malden: Blackwell Publishers, 2008), pp. 319–33. He introduces terms such as 'distinctification' on p. 325.

[15] Nelson (2008), p. 320ff.

[16] Descartes speaks of primitive notions in a 21 May 1643 letter to Elizabeth (AT III 665; CSMK III 218).

[17] AT VIIIA 25; CSM I 210.

God, mind, body (and on Nelson's account, union).[18] These, if I have Nelson's view right, are *the* innate ideas. Extension, then, serves to make 'objects' like the triangle (a mathematical object) and the sun (an empirical object) intelligible.

When analysing our ideas so as to reveal their underlying objects we are engaged in what Nelson calls the process of *distinctification*.[19] So, in following the analysis of certain ideational elements out to the idea's true underlying object, which we know will be either infinite mind (God), finite mind, or extension (as I said, Nelson's account includes union), we are 'distinctifying' the idea. The route one takes in bringing to cognitive light the idea's underlying object (via the procedure of distinctifying) is what Nelson calls a 'cognitive route'. It cannot be stressed enough that the cognitive routes lead to at least one of the three (and on Nelson's account, four) innate ideas—God, mind, body (and union). These four innate ideas lie at the bottom, so to speak, of all other ideas. As Descartes puts it, they are 'the patterns on the basis of which we form all our other conceptions',[20] standing independently (cognitively speaking) of one another, each being 'understood only through itself'.[21] Thus, we cannot use the primitive notion of body to make intelligible the primitive notion of mind, and vice versa. The primitive notions are the *principles* of philosophy.

The above reading suggests a reply that was available to Descartes—a reply to those who, recall, claimed that Descartes's matter did not have any ontological bearing on the extra-mental status of sensible matter. Recall that his objectors concluded that his physics failed to tell us anything about a sensible, material reality. To be sure, matter (*res extensa*) does not exist independent of God, but it can and does exist independent of finite mind.[22] How is this shown? We can conceive of mind without ever conceiving of body, and vice versa. That is, we can *exclude* extension from our idea of mind, and *thought* from our idea of body. Even so, once clear and distinct, we cannot conceive extension as not existing (or as not enduring), since the two attributes (extension and duration) are only conceptually distinct. We can focus on one while *ignoring* the other, but we cannot exclude one from the other. Why is this? God. His conceiving matter creates it. Thus, corporeal substance exists (or endures) independent of finite mind (the thinking substance). So, if Descartes's objectors come to clearly and distinctly conceive extension, which will include their understanding that extension is really distinct from thought, they will not be able to coherently doubt its extra-mental existence. So, if Descartes's physics is rooted in this extension, his physics in fact will tell us about an extra-mental material reality.

But even this, as close as it is to being right, is not exactly right. What we need to see, I contend, is that what was called *sensible matter* is categorized as empirically real. Given that what was called *intelligible matter* is taken to be that which underwrites the very possibility of empirical reality, it follows that sensible matter will presuppose intelligible matter. This, I think, is really the view that is lurking. And so, if Descartes's physics is rooted in intelligible matter—that is, in the very condition underlying empirical reality, then his physics *could not fail* to tell us about a mind-independent material reality—where by the latter I mean *empirical reality*. This, I think, looks similar to the sort of thing that Kant would eventually say about the status of both natural law-propositions and mathematical propositions—namely, that even though they are *synthetic*, they are nevertheless *a priori*, insofar as they transcend any *particular* moment of sensory experience (that is, any instance of what Kant called *intuition*), serving as the condition for the very possibility of the sensible.[23]

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[18] Nelson (2008) includes union in the list on p. 328.

[19] Ibid., p. 325.

[20] AT III 665; CSMK III 218.

[21] Ibid. He also says that they cannot be understood in terms of any of the others.

[22] AT VIIIA 24–5; CSM I 210.

This is where I believe Nelson's reading looks to tie into Friedman's remark about the objective point of view, mentioned at the opening of this chapter, where Descartes is led to talk about matter's being established by way of divine perspective. It is God's point of view that *guarantees* matter's existence. We might be tempted to backslide here and say that, as is the case with mathematical truths, extension is *necessary* (that is, it exists necessarily). But, as Nelson and Cunning would remind us, even though extension exists (and perhaps must, since God conceives it), it is nevertheless only a possible existent. How so? It depends ontologically on God. Thus, it seems that for Descartes the conditional—*if mathematics is intelligible, matter exists*—could not be rendered false. It cannot because, in including God among those who conceive extension (in making mathematics intelligible), Cartesian metaphysics guarantees that God creates extension, thus guaranteeing the truth of the conditional.

Consider this in light of the divided line. The first argument for the existence of matter, discussed in the previous chapter, which as we saw builds on the idea that matter (or corporeal substance) is the *cause* of sensation, should be understood as emphasizing the move from Plato's category of images to the category of objects *qua* originals, where my sensations are items in the former category (my sensations depend in an important sense on *my* mind), and body an item in the latter category. Matter (body or *res extensa*), in this second category, is an object *qua* original. The view of objects as existing independent of mind is in part captured in Friedman's talk about the objective point of view. As I noted, this seems to be akin to the sort of view that Moore established in his analysis of Kant's phrase 'to be met with in space'. Chappell's, Nadler's, and Hoffman's respective views were taken to be attempts at working out Descartes's view of body (or matter) in exactly the sense of 'an objective point of view', namely, as being empirically real. What I am wondering here is whether Lennon, Nelson, and Nolan should be read along similar lines. To remind the reader of my aim, the answer to this will be a resounding, 'no'.

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[23] Kant says this, for example, in the First and Second Part of the *Prolegomena*. Immanuel Kant, *Prolegomena To Any Future Metaphysics*, translated by James Ellington (Indianapolis: Hackett Publishing, 2002).

Now, once Descartes established the concept of empirical reality, Friedman sees Descartes as going on to ask about what underlies our even being able to make the distinction between the empirically ideal and empirically real (or between the subjective and objective points of view).[24] This sort of inquiry is, I think, in turn connected to Friedman's insight that suggests a possible second and equally important argument for the existence of matter. In looking at what underlies our being able to even conceive of the empirically real (versus the empirically ideal), we are engaged in what Kant calls 'philosophical reflection', and subsequently engaged in doing some early version of transcendental philosophy. That Descartes's own project can be cast in this light may be of concern to some scholars. But, let's see what the texts suggest. There are two possible readings here. The first tells us that extension emerges in the analysis as part of the account of transcendental ideality. The second tells us that extension emerges in the analysis as part of the account of transcendental reality. I will look briefly at both readings.

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[24] Allison (1983) refers to the latter as 'inner-subjective'.

16

Transcendental Matter

Although Friedman focuses primarily on Newton's philosophical view of matter, he suggests that Kantian-like philosophical reflection is to be found in Descartes.[1] Friedman lays out the line of reasoning employed by Kant in showing the necessity of the Newtonian law of universal gravitation. There are at least three steps:

1) That which agrees with the formal conditions of experience (according to intuition and concepts), is *possible*.
2) That which connects with the material conditions of experience (sensation), is *actual*.
3) That whose connection with the actual is determined in accordance with universal conditions of experience, is (exists as) *necessary*. (A218-219/B265-266)[2]

The move, says Friedman, progresses 'from *possibility*, through *actuality*, to *necessity*'.[3] He says something similar in his earlier analysis of Descartes and the argument for the real existence of Cartesian matter: 'What the argument of the sixth *Meditation* then adds, of course, is precisely the real or actual existence of what is so far merely possible.'[4] So, the Sixth Meditation proof employs steps (1) and (2) above. Necessity of matter (extension), on Descartes's view, seems to be derived from the truths of mathematics, which is in turn connected to God's determining that necessity (by willing or conceiving extension in his making mathematics intelligible). So, Descartes's proof could also be understood as progressing from possibility, to actuality, to necessity.

As was pointed out in the last chapter, the sun or *any* body conceived in exclusion of extension is unintelligible. Extension, as I have been suggesting, is underwriting the intelligibility of the empirically real. Friedman appears to be arguing that the transition Descartes is making, in casting extension as the underlying condition for the intelligibility of an extra-mental material world, is a move that no longer deals with extension as something found in the category of the empirically real, but deals with extension as something found in the category of the *transcendentally ideal*. In Plato's terms, this would appear to be moving from thinking of objects *as* originals to thinking of those same objects now *as* images. So, the move here is from the Visible section of the line to the Intelligible section. Let's look at two possible readings of Descartes.

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[1] Michael Friedman, 'Transcendental Philosophy and Mathematical Physics', *Studies in History and Philosophy of Science*, 34 (2003), pp. 29–43.

[2] Ibid., p. 33. The references 'A218–219/B265–266' are to Kant's *Critique of Pure Reason*.

[3] Ibid.

[4] Ibid., p. 154.

EXTENSION AS TRANSCENDENTALLY IDEAL

Some of the most compelling textual evidence that suggests Descartes's having some grasp of what we are calling the *transcendental ideality-reality distinction* (the distinction that appears to be similar to Plato's Intelligible section of the line) is found in his discussions of the principal attributes and the intelligibility of substance. Rehearsing some of what was established earlier, though now going into things in more detail, recall *Principles*, Part I, Article 53, which we considered in the last chapter:

> A substance may indeed be known through any attribute at all; but each substance has one principal property which constitutes its nature and essence, and to which all its other properties are referred. Thus extension in length, breadth and depth constitutes the nature of corporeal substance; and thought constitutes the nature of thinking substance. Everything else which can be attributed to body presupposes extension, and is merely a mode of an extended thing.[5]

The 'presupposes' here is important. As he goes on to say:

> For example, shape is unintelligible except in an extended thing; and motion is unintelligible except as motion in an extended space . . . By contrast, it is possible to understand extension without shape or movement.[6]

So, the mode *shape* is unintelligible independently of extension. Yet, extension (generally conceived) is intelligible independently of shape. As in Part II, I will take this to express logically or conceptually the following relation: shape *entails* extension. In terms of 'presupposes', shape presupposes extension. Consider this in light of what he says in *Principles*, Part I, Article 52:

> if we perceive the presence of some attribute, we can infer that there must also be present an existing thing or substance to which it may be attributed.[7]

Here, in line with Nelson's reading, we get the claim that extension *qua* attribute *entails* existing substance. Again, by using the 'presupposes' language, Descartes is saying that the idea of extension presupposes the idea of existing substance. This is confirmed by our earlier analysis of Descartes's theory of attributes, conceptual distinction, and the doctrine of divine simplicity. Gathering together what we know, then, we can make the following argument: if we clearly and distinctly perceive shape, we *ipso facto* perceive extension (Part I, Art. 53), and if we clearly and distinctly perceive extension, we *ipso facto* perceive existing substance (Part I, Art. 52).

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[5] AT VIIIA 25; CSM I 210. [6] Ibid. [7] Ibid.

The idea of shape-entailing-extension-entailing-substance is what Descartes calls a 'complete' or 'adequate' idea of a corporeal substance (sometimes he refers to it as the idea of a complete substance or entity).[8] So, if in my examination of my sensory idea of the sun I am able to render the idea (via analysis, say) so that it reveals this structure (shape-entailing-extension-entailing-substance), I can be said to have a clear and distinct idea of a corporeal substance. Let's say that I do the same thing with my sensory idea of the moon, producing the clear and distinct idea of a corporeal substance. Does this show that I clearly and distinctly perceive *two* corporeal substances? No.[9] If I take the first idea and compare it to the second, the distinction that I will draw between the two 'objects' will be a modal one. For example, the sun will be greater in size than the moon. The essence of both is extension. So, they do not differ in this respect, and I cannot appeal to their essences to draw any distinction between them.[10] We found this in Nolan's view.

As I stated earlier, the above reading (informed by Friedman, Lennon, Nelson, and Nolan) shows Descartes to be a transcendental idealist. Let me explain. First, we should recall that Descartes says that space is identical with extension *understood as* length, breadth, and depth (this is body taken in its general sense).[11] And, he says, 'when time is distinguished from duration taken in the general sense and called the measure of movement, it is simply a mode of thought'.[12] As Geoffrey Gorham has made clear, time is a way of regarding the duration of motion.[13] This is related, he argues, to Descartes's saying that number or quantity is a way of regarding extension.[14] This is important, and is aligned with some of what we have already considered. When Descartes is talking about ways of regarding, extension (here, cast as *space*) and time, I contend, seem to be functioning as underlying conditions of the intelligibility of individuated bodies. Let me make this clearer.

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[8] See Kurt Smith and Alan Nelson (forthcoming 2009).

[9] I am indebted to Cathay Liu (UNC Chapel Hill philosophy) for suggestions here.

[10] This interpretation of Descartes looks to have its origin in Spinoza. Benedict Spinoza, *Ethics*, translated by A. H. Stirling (Hertfordshire: Wordsworth Editions, 2001), p. 5; Part I, Proposition V.

[11] AT VIIIA 46; CSM I 227.

[12] AT VIIIA 27; CSM I 212.

[13] Geoffrey Gorham, 'Descartes on Time and Duration', *Early Science and Medicine* 12 (2007), pp. 28–54.

[14] Gorham (2007), pp. 35–6.

I will do this by first asking: *how* is extension being employed by Descartes in the above account of the intelligibility of number? What is involved (conceptually) in saying that an object is made intelligible via extension, an object such as a number, or a triangle, or the sun? Lennon, Nelson, and Nolan locate individuated bodies, such as the triangle or the sun in the mind (as acts of regarding), and yet they see the need (in interpreting Descartes) to assign the role of 'intelligibility maker' to extension (to *res extensa*) when it comes to the ideas of bodies. This is where things get tricky. As Kant argues, if, when inquiring into the conditions underlying the possibility of empirical reality, we take ourselves to be inquiring into how our perceptions of bodies correlate with the bodies as they exist (or the one body, *res extensa*, as it exists) independent of the mind, we 'have misapplied our concepts'.[15] In other words, if we think that we must choose between either mind or body here—in accounting for the possibility of our *experience of bodies*, we suffer from the false impression that we are working within the conceptual framework of empirical realism. As he puts it: 'dualism [holds] *only* in the empirical sense'.[16]

By contrast, when thinking within the conceptual framework of a transcendentalist view, Kant says:

> Neither the *transcendental object* which underlies outer appearances nor that which underlies inner intuition, is in itself either matter or a thinking being, but a *ground* (to us unknown) of the appearances which supply to us the empirical concept of the former as well as the latter mode of existence.[17]

When Lennon, Nelson, and Nolan appeal to extension as that which underlies the intelligibility of individuated bodies, such as the sun or a triangle, their doing so, it seems to me, is driven by a misapplication of the concepts—that is, if they take extension to be something that we *directly* perceive. My sensory idea of the sun, for example, is not, on their view, of the sun at all—it is *really* of extension. This is what I *really* see when surveying the room before me. The tables and chairs that *appear to me* are simply ways of regarding this object of perception (extension), these 'bodies' (plural) being nothing more than inventions of my mind. Lennon, Nelson, and Nolan appear, then, to take themselves as offering an empirical realist view of body (the one body as opposed to those other empirical realists who advocate for the 'many bodies' view). But if so, as Kant's argument above suggests, their views suffer from a sort of category mistake. I want to suggest that their views are best understood as offering an account of the very *ground* of empirical realism. As I suggest earlier, they are tracking a transcendental strain of thinking in Descartes about the reality of matter. This could be made clearer.

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[15] Kant, *Critique of Pure Reason*, A 379 (Norman Kemp Smith), p. 352.

[16] Ibid., A 379 (Norman Kemp Smith), p. 351. Italics and bracketed remark are mine.

[17] Ibid., A 380 (Norman Kemp Smith), p. 352. Emphasis is mine.

Let me do this by now asking: what happens to *the sun*—as a way of regarding extension, I mean—if there is no extension? According to Lennon, Nelson, and Nolan, the sun is rendered unintelligible. But, this very way of putting it, I think, misleadingly suggests the picture of a mind being 'directed at' extension, and in regarding extension a certain way (this is all part of the mind's act of directing itself at extension), the mind is made aware of *res extensa* by way of regarding it *sun-ly* (when having the idea of the sun). The question (at the top of the paragraph) is raised within the conceptual framework of empirical realism (where all we have done is exchanged the sun for extension). The above question and answer suggest, then, that we render the sun unintelligible by simply removing extension—as though the mind would still be *doing* something (though *what* that would be is a mystery), but now only without extension as its object (the mind, now intellectually 'blind' as though the lights had been turned out). What I want to suggest is that extension is not something we *ever* perceive—if we are talking within the conceptual framework of empirical reality. On the empirical realist reading, when perceiving we perceive individual bodies such as the sun. Extension (*res extensa*) is not an object of the senses—it is, as Descartes carefully argues (in the Second Meditation, for example), an intellectual entity (an object of intellection).

As we saw earlier, Lennon, Nelson, and Nolan take Descartes's theory of attribution to be internally connected to his theory of conceptual (or rational) distinction. An attribute is only conceptually distinct from the substance of which it is the attribute. What is more, if I understand their views correctly, an attribute is—and this is important—the way a substance is *given to us* in thought. As Descartes puts it, a substance is 'known *through*' an attribute.[18] One way to apply this to the view that ideas are *ways of regarding*, is to recast the *attribute* as the way of regarding. I am suggesting, at the risk of going against the way Lennon, Nelson, and Nolan read their own views, that *the sun* is the object of perception, and it is the *attribute* (extension, here divided by motion) that is the way of regarding. In Kantian terms, we might say that space and time per se are not what we think *about*, rather they are the *form* that our regarding takes when perceiving such objects as the sun. The *presentational form* of the overlying object (which is an *act* of ideation) is where we ontologically locate spatiotemporality. It is *that by way of which* objects, such as the sun and moon, are presented to (or by) the mind. The same sort of thing can be said of the idea of the triangle.

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[18] AT VIIIA, 25; CSM I, 210.

Given that the individuating of bodies is rooted in the act of regarding (where the form of regarding is spatiotemporal—in Descartes's terms this is extension divided by motion—the perceived duration of motion, recall, is time), Descartes's view seems to be that individuated bodies are in part conceptual entities,[19] or, as Lennon suggests, *phenomenal* entities,[20] which does more than to suggest the Kantian move to the transcendental conceptual framework. Time is brought about as a way of regarding the duration of motion. And, even though ideal (as Lennon argues) it is motion that accounts for any 'division' of extension (*res extensa*). Space (extension regarded generally) divided by motion, then, looks to be the very condition necessary for the very *possibility* of 'individuated' bodies—that is, for things that are empirically real.[21] And, given that this is within the context of *conditions-of-conceivability*-talk, space and time (that is, extension divided by motion) appear, as I said, to be functioning along the lines of Kant's forms of intuition.[22] That which is not subject to such conditions, the bare substance (corporeal substance minus the attribute extension), is unintelligible. This is akin to Kant's noumenon. Thus, in casting extension divided by motion as the underlying condition for the possibility of what we are calling empirical reality, it would appear to be something classifiable in the category of the *transcendentally ideal*, where the bare substance (that is, corporeal substance minus *all* attributes—or, minus the various acts of regarding), would be something classifiable in the category of the transcendentally real. So, on this interpretation, where Descartes locates the ultimate conditions of the conceivability of the empirically real in the category of transcendental ideality (for Plato it would be the category of Objects *qua* Images), Descartes is cast as a transcendental idealist.

Let's put this in the context of the divided line. Start with the distinction between images and objects *qua* originals, or in Kantian terms, the distinction between empirical ideality and empirical reality. The sun in the heavens is taken to be the cause of the sensory image of the sun. The image is empirically ideal (the objective sun), the sun in the heavens is empirically real (the formal sun). Now, recasting the formal sun in terms of its conceivability, we discover (I am referring to what we just said about Descartes) that the underlying condition of intelligibility of any empirically real object (such as the formal sun) is extension (space) divided by motion (time). Independent of divided extension is the substance, but if extension (divided by motion) is excluded, the substance is rendered unintelligible. Thus, substance (minus attribution), as I said, falls into the category of transcendental reality. The divided line might look as follows:

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[19] Sowaal (2004) sorts out Descartes's terminology with respect to 'conceptual entity'-talk.

[20] Lennon (2007), p. 30ff.

[21] For more on this, see Sowaal (2004).

[22] For more on Kant, see Corey Dyck's discussion in his 'Empirical Consciousness Explained: Self-Affection, (Self-) Consciousness and Perception in the B Deduction', *Kantian Review*, vol. 11, 2006, pp. 29–54. I also thank him for a stimulating discussion related to this material during my visit to the University of Western Ontario.

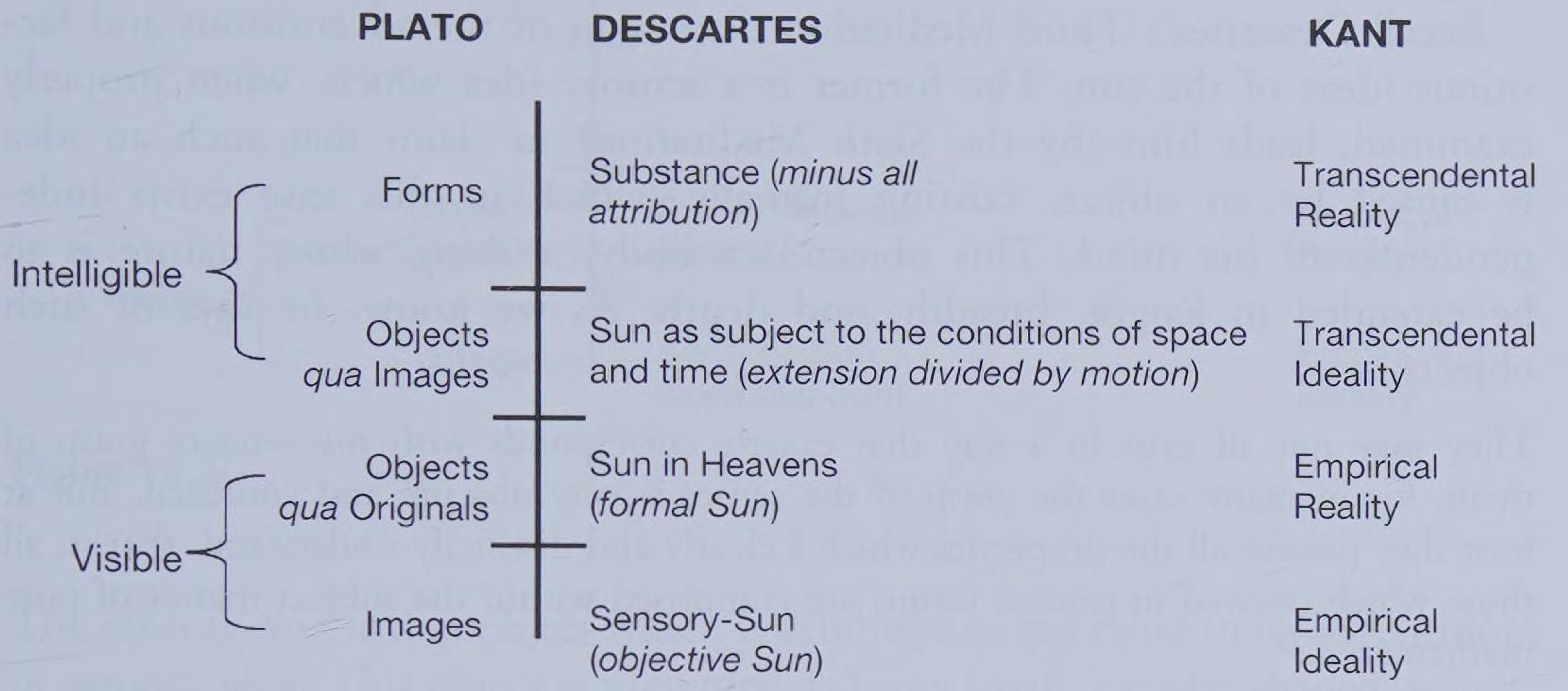
PLATO
DESCARTES
KANT
Intelligible
Forms
Substance (*minus all attribution*)
Transcendental Reality
Objects *qua* Images
Sun as subject to the conditions of space and time (*extension divided by motion*)
Transcendental Ideality
Visible
Objects *qua* Originals
Sun in Heavens (*formal Sun*)
Empirical Reality
Images
Sensory-Sun (*objective Sun*)
Empirical Ideality

Figure 16.1

Matter (*res extensa*) is located in the category of the transcendentally ideal. Thus, on this reading, in respect to matter Descartes looks to be a transcendental idealist. And, even though matter is taken to be transcendentally ideal—so that there is really only *one* body (there is simply 'space' or extension serving the metaphysical function)—this way of reading Descartes nevertheless leaves room for the empirical realist reading, which claims that there are *bodies*, such as the sun, that exist independent of the subjective instances of sensory perception.

EXTENSION AS TRANSCENDENTALLY REAL

Another way to read Descartes puts Cartesian extension in the category of the transcendentally real. If this turns out to be his position, we can understand why Kant took Descartes to have been seriously confused. This is so because, in categorizing extended substance as transcendentally real, Descartes would be claiming that extension (extended substance) can exist independently of the conditions of space and time. However, since extension is *identical* to space (the two are only conceptually distinct),[23] it follows that if Descartes put extension in the category of transcendental reality, he would be claiming that extension can exist independently of extension, which is absurd. Even so, there is a way of reading Descartes where he nevertheless puts extension in this category.

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[23] AT VIIIA 45–6; CSM I 227.

Recall Descartes's Third Meditation discussion of the adventitious and factitious ideas of the sun. The former is a sensory idea which, when properly examined, leads him (by the Sixth Meditation) to claim that such an idea is caused by an object, existing formally, which in this case exists independently of his mind. This object is a body, a thing whose nature is to be extended in length, breadth, and depth. As we know, he says of such objects:

> They may not all exist in a way that exactly corresponds with my sensory grasp of them, for in many cases the grasp of the senses is very obscure and confused. But at least they possess all the properties which I clearly and distinctly understand, that is, all those which, viewed in general terms, are comprised within the subject-matter of pure mathematics.[24]

The constructed or factitious idea of the sun exhibits the sun as a geometrical or mathematical object. It is not the object presented to me in sensory experience, but this is not a problem once I have undertaken a philosophical examination, as worked out in the *Meditations*. For, I will know which properties I need to ignore (and subsequently exclude) when constructing the idea of the sun that depicts it as a mathematical object (a sphere much greater in size than the earth, *minus* colour, heat, and so on). As we know from Part II, the sensory idea of the sun is obscure and confused because (1) there is no intelligible relation holding between the elements of the idea (for example, there is no relation holding between the *circular* shape presented in the idea and the colour *yellow*); and (2) the elements of the idea come from different classes in the enumeration (the element *circular* is ultimately a member of *E*, the element *yellow* is ultimately a member of *T*). By contrast, when we focus on those elements that are members of *E*, which are comprised within the subject-matter of mathematics (for example, *shape* and *size*), the new idea (the constructed factitious idea), is clear and distinct. This is so because (1) there is an intelligible relation holding between the elements of the idea (*shape* and *size* are related mathematically); and (2) the elements of the idea come from the same class of the master enumeration, namely *E*.

If we follow Friedman's lead, as I have done thus far, Descartes appears to be making two distinct claims about matter. Let's rehearse some of what has been established before moving forward. First, as we know, matter is empirically real. This is established in part by the Sixth Meditation proof for the existence of body. So, just as we did above, we might see the move as follows:

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[24] AT VII 80; CSM II 55.

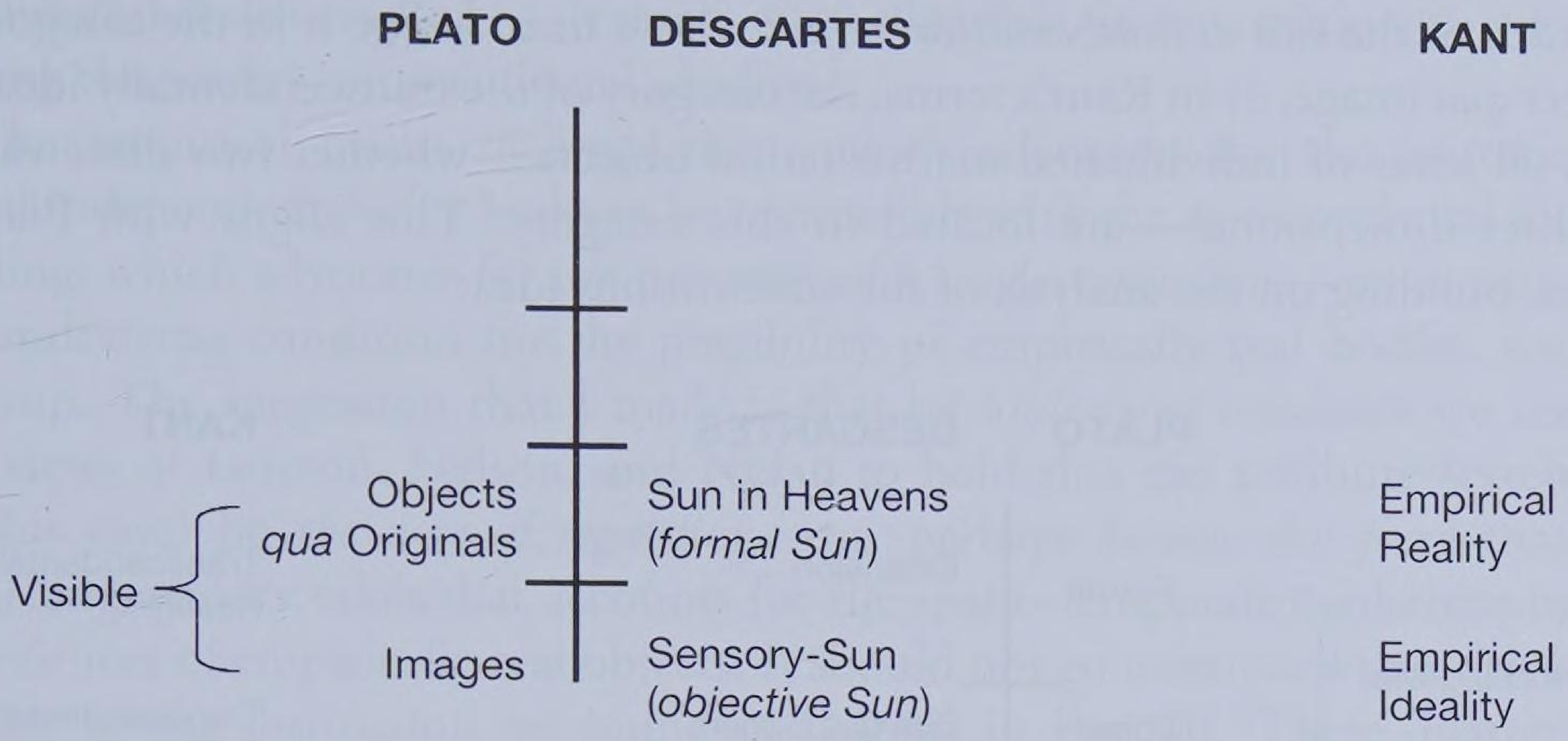

Figure 16.2

The sensory-Sun is that object which is exhibited to the mind in its adventitious or sensory idea. This object is presented as being small, circular-shaped, yellow, hot, and so on.[25] On this reading (as with the other readings considered thus far), I place this in the image category, or in Kant's terms, the category of empirical ideality. Descartes says that he has always believed that such objects (sensory images) originated 'from things which are located outside me',[26] where he has had a strong impulse to believe that 'there exists things distinct from myself which transmit to me ideas or images of themselves through the sense organs or in some other way'.[27] By the Sixth Meditation, as we now know, this impulse or great propensity must be trusted, for God is its origin and 'I do not see how God could be understood to be anything but a deceiver if the ideas were transmitted from a source other than corporeal things.'[28] 'It follows,' he says, 'that corporeal things exist.'[29] So, material objects, such as the sun, are empirically real.

Descartes suggests (in light of the Third Meditation) that although he takes the sun, for instance, to be the origin of his sensory idea of it, the sun as extra-mental object will not possess many of the properties it is exhibited as having in his sensory idea. For example, it will not be yellow or hot. Rather, it will possess certain mathematical properties, such as shape, size, motion, and so on. We are now in a position to construct the factitious idea of the sun, which is the idea that represents the sun as being a sphere much greater in size than the earth. This idea, says Descartes, better represents (in the sense that it better resembles) the sun than does the sensory or adventitious idea.[30] Arguably, it better resembles the sun insofar as it depicts the sun as a system of motion. Be that as it may, it is the idea that allows the natural philosopher to deal with the sun within the context of an astronomical (mathematical) science. Here, I want to suggest that

[25] AT VII 39; CSM II 27.
[26] AT VII 38; CSM II 26.
[27] AT VII 39; CSM II 27.
[28] AT VII 80; CSM II 55.
[29] Ibid.
[30] AT VII 39; CSM II 27.

the idea of the sun *as mathematical object* allows us to locate it in the category of object *qua* image, or in Kant's terms, the category of the transcendentally ideal. In fact, all ideas of individuated mathematical objects—whether two-dimensional or three-dimensional—are located in this category. This aligns with Plato.[31] Thus, building on the analysis of the adventitious idea:

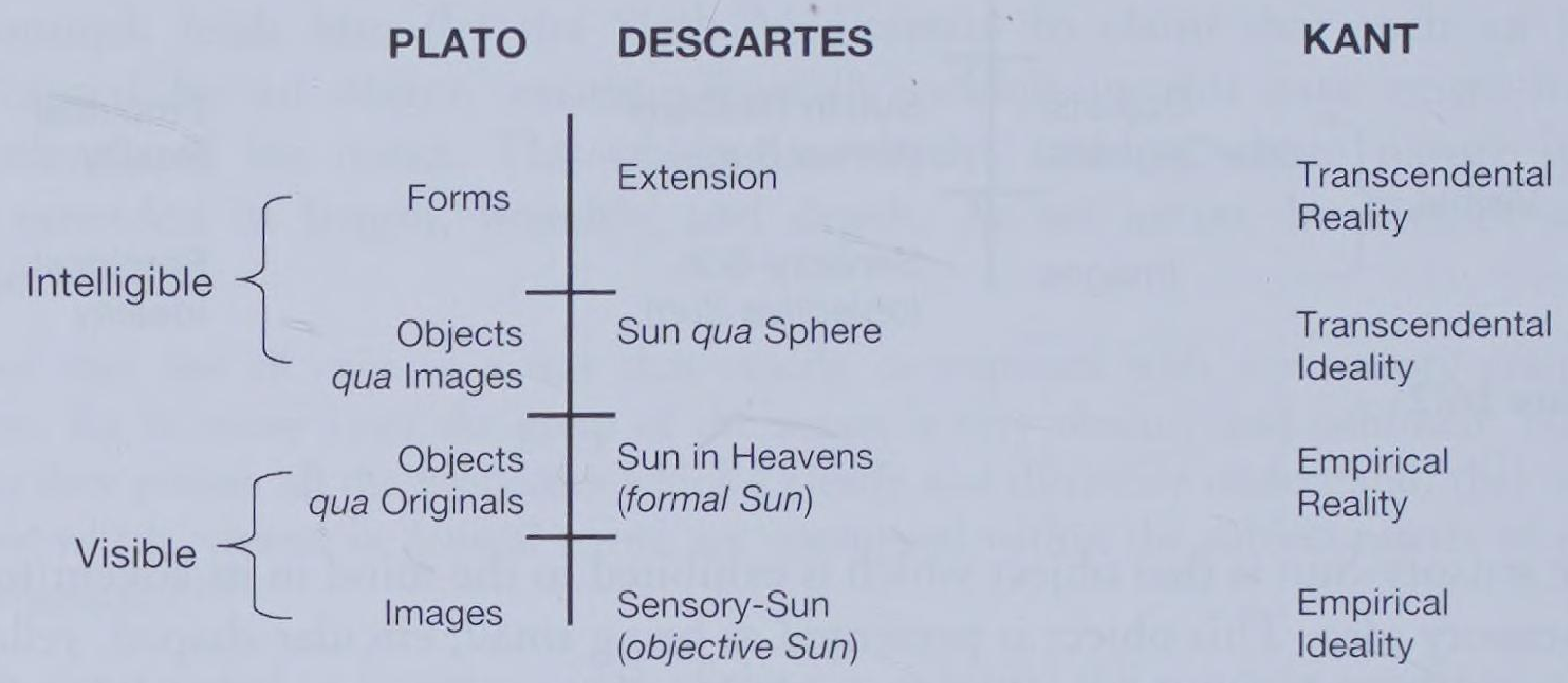
PLATO
DESCARTES
KANT
Intelligible
Forms
Extension
Transcendental Reality
Objects *qua* Images
Sun *qua* Sphere
Transcendental Ideality
Visible
Objects *qua* Originals
Sun in Heavens (*formal Sun*)
Empirical Reality
Images
Sensory-Sun (*objective Sun*)
Empirical Ideality

Figure 16.3

Extension, on this reading, is what underwrites the possibility of such things as spheres and cubes. Clearly, in taking the transcendentally real to be *in*conceivable (because it is not subject to the conditions of space and time), Kant would disagree with Descartes's categorizing extension as transcendentally real—just as he would disagree with Plato when categorizing the Forms here, the Forms taken by Plato as being *the* intelligible objects. Even so, on this interpretation, Descartes, like Plato, is cast as a transcendental realist.

Unlike the transcendental idealist reading, the transcendental realist reading leaves little room, if any, for the empirical realist reading. As Kant says,

> Transcendental realism . . . inevitably falls into difficulties, and finds itself obliged to give way to empirical idealism, in that it regards the objects of outer sense as something distinct from the senses themselves, treating mere appearances as self-subsisting beings, existing outside us.[32]

So, if Descartes held some form of transcendental realism, he was very confused, because it would ultimately put him in conflict with his empirical realist view of matter. I leave it to scholars to decide whether Descartes was ultimately a transcendental realist. From the sketches of both readings—transcendental idealist and realist readings—it seems more plausible, in my opinion, to put him among the (early) transcendental idealists.

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[31] I am indebted to Tom Lennon, Henrik Lagerlund, and Corey Dyck who, at a talk at the University of Western Ontario, convinced me that one way to read Descartes is to see him as categorizing the ideas of individuated geometrical figures as transcendentally ideal, where extension would then be categorized as transcendentally real. In a recent article, Nolan (2005) also places mathematical objects in what I call the transcendentally ideal category.

[32] Kant, *Critique of Pure Reason*, A 371 (Norman Kemp Smith), p. 347.

The empirical realist reading, then, which advocates for the existence of mind independent *bodies* looks to be compatible with the transcendental idealist reading, which advocates for the necessity of a single extension—*res extensa*—as an underlying condition for the possibility of empirically real bodies, such as the sun. The suggestion that I made is that by *underlying condition* we rework the views of Lennon, Nelson, and Nolan to hold that the attribute (extension in this case) be *the way of regarding*—or, perhaps better, the *form* that the regarding activity takes that accounts for the spatio-temporality inherent in our experiences of empirically real objects. It should not go unnoticed that this looks similar to the distinction we found in Leibniz in Part III. There, matter was taken to be metaphysically prior to individual bodies. So, it seems that for both Descartes and Leibniz this extended thing (matter) is necessary in the sense that it underwrites the very intelligibility (or possibility) of empirical reality. In following Friedman as I have done here, we are able to accommodate what (before our examination) appeared to be opposing views. My assessment, then, is that the views of Chappell, Hoffman, and Nadler track a line of reasoning in Descartes that emphasizes matter's being empirically real. The views of Lennon, Nelson, and Nolan, on the other hand, track a different line of reasoning: one that emphasizes matter's being either transcendentally ideal or transcendentally real.

If this is right, then the 'many bodies' view (the empirical realist reading) and the 'one body' view (the transcendental idealist reading, for example) need not be *competing* views, in the sense that from the different conceptual perspectives Descartes could have held *both*. And it is possible to hold both. As Kant himself puts it:

> The transcendental idealist . . . may be an empirical realist or, as he is called, a *dualist*; that is, he may admit the existence of matter without going outside his mere self-consciousness, or assuming anything more than the certainty of his representations, that is, the *cogito, ergo sum*.[33]

In another place, Kant says that in keeping the distinctions straight (the empirical and transcendental ideality/reality distinctions), and in undertaking the right sort of philosophical investigation, one will be led to holding both transcendental idealism and empirical realism:

> The transcendental idealist is, therefore, an empirical realist, and allows to matter, as appearance, a reality which does not permit of being inferred, but is immediately perceived.[34]

Lennon, Nelson, and Nolan bring to the surface precisely the internal struggle in which Descartes is engaged between, what I think are, his empirical and transcendental conceptions of matter.[35] With this struggle in mind, let me reiterate my proposal. In light of the conception of empirical reality, Descartes held there to be many individual bodies; in light of the conception of transcendental ideality, for example, he held there to be only one body, the underlying condition for the intelligibility of empirically real bodies, namely, *res extensa.* This extension (here functioning along the lines of Kant's form of intuition) informs ideational activity (specifically the act of representation), accounting for the very intelligibility (or possibility) of what we called the empirically real world. We should see the sophisticated and compelling readings considered above (Nadler's to Lennon's), then, as working together to form a richer interpretation of Descartes.

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[33] Ibid., A 370 (Norman Kemp Smith), p. 3.

[34] Ibid., A 379 (Norman Kemp Smith), p. 347.

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[35] Lennon says as much in Lennon (1986).

Tying up the Loose Ends—Closing Remarks

In Part II, recall, it was noted that in *The World* Descartes says that the eternal truths, such as *the whole is greater than its parts*, underwrite mathematics.[1] Add to this something he says to Clerselier in the Appendix of the Fifth Set of Objections and Replies. There, he again (not surprisingly) counts such propositions as *when equal quantities are taken from equal amounts the remaining amounts will be equal*, and *the whole is greater than its parts*, to be among the 'elements of geometry'.[2] The latter proposition is of importance to my view, especially in light of a 27 May 1638 letter in which Descartes says to Mersenne:

> You ask whether there would be real space, as there is now, if God had created nothing. At first this question seems to be beyond the capacity of the human mind, like infinity, so that it would be unreasonable to discuss it; but in fact I think that it is merely beyond the capacity of our imagination, like the questions of the existence of God and the human soul. I believe that our intellect can reach the truth of the matter, which is, in my opinion, that not only would there not be any space, but even those truths which are called eternal—as that 'the whole is greater than its part'—would not be truths.[3]

On the face of it, the import of this remark is not clear. Initially, without considering other passages, it could be read as simply telling us that had God not created the cosmos, items such as space and mathematical truths would not exist—or something to this effect. As I suggested back in the Introduction and (more forcefully) in Part II, Descartes's mention of both space and mathematical truths here is not to be taken lightly—he is not simply offering an arbitrary list of items that would not exist had God not created the cosmos. In light of Cartesian metaphysics, there is no doubt that by *space* Descartes means corporeal substance—it is nothing short of *res extensa* itself.[4] The question that Mersenne raises in his letter, according to Descartes's reply, was about whether space was something that God needed to create (if it is a *nothing*, why must we take it to be a *something* that God creates?). That Descartes includes the reference to mathematical truths here is, I contend, significant. For, he seems to be suggesting that God's not creating space (that is, Cartesian matter) entails

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[1] AT XI 47; CSM I 97.

[2] AT VII 206; CSM II 271. CSM notes that these are Axioms found in Euclid's *Elements* (CSM II 271, fn. 2).

[3] AT II 138; CSMK III 103.

[4] AT VIIIA 45; CSM I 227.

God's not creating other items such as mathematical truths. The two—matter and mathematics—are related, where the relation seems to be that mathematical truths depend in some ontological sense on the existence of matter, such that if God fails to create the one, he fails to create the other—they depend on matter insofar as, excluded from it, they are rendered unintelligible. If this is right, what Descartes appears to be saying in this passage is that had God not created *res extensa* (or as I have been calling it, matter), there would be *no* mathematical truths.

So, reconsider the conditional I set out to better establish in Part IV:

If mathematics is intelligible, matter exists.

If my reading of the above remark to Mersenne is right, Descartes seems to be saying that if we deny the consequent of this conditional, we must deny its antecedent. And so, if matter does not exist, mathematics is unintelligible.

A reconsideration of other claims that we have entertained can now be seen in support of this reading. For example, reconsider Descartes's claim that even though we need not refer the quantity *ten feet* to this or that determinate hunk of extension, we must refer it to *some* (hunk of) extension, otherwise the quantity is rendered unintelligible.[5] Here it again appears that Descartes is committed to the inference that were there no matter, mathematics (here mathematical entities like the magnitude *ten feet* and the number *ten*) would be unintelligible. Descartes could also remind his objector that the primary reason why minds cannot engage in mathematics, given that there is no matter, is that there is only a *conceptual* distinction between quantity or number and extended substance.[6] So, if the concern is that there can be numbers (intelligible mathematical entities) and yet no corporeal substance, the trouble with the objector's view is that it assumes that number and extension are *really distinct*—since his objector takes the one to be able to exist without the other.[7] That is, such an objector would confusedly hold that he could conceive of number while *excluding* extension. And so, given that number and matter are only conceptually distinct, our objector is demonstrating that he is deeply confused in taking them to be really distinct.[8] To be sure, the mathematician could be said to be *ignoring* extension or extended substance *while* doing mathematics (where mathematics involves a kind of abstraction), but for all that, extension is nevertheless something that is underwriting the intelligibility of the object (a number or shape) with which the mathematician is dealing, and so it is not something that he can *exclude* from number.

Another reason for Descartes's holding the *mathematics-requires-matter* view is his thinking that the sort of divisibility one needs for mathematics (the sort that produces quantity or number) requires corporeality—for whereas we cannot conceive of matter or extended substance 'except as being divisible',[9] we cannot conceive of mind or thinking substance 'except as being *in*divisible'.[10] *Divisibility* is an underlying condition of mathematics. This is something that other mathematicians would also come to emphasize in number theory.[11] Recall from Parts I and II that Aquinas took there to be two kinds of division: the division of a material substratum that resulted in numbers, and the division of an immaterial substratum that resulted in the 'plurality' of the forms. But here, again recall, the plurality associated with the forms (or transcendentals) was *not* quantitative in nature. Number simply fell out as meaningless in respect to this kind of division. It was, we noted, something like a category mistake to use *number* in this context. If divisibility turns out conceptually to be a necessary condition for mathematics, and mind is in principle indivisible, and talk about a plurality of minds is non-quantitative (non-number) talk, then thinking substance alone cannot secure the conditions necessary for mathematics—so, given a world in which there is no matter, there is no medium by way of which number is produced (this would be coextensive with its being cognized).

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[5] AT VIIIA 44; CSM I 226.

[6] Ibid.

[7] I suspect that our objector could take number to be modally distinct from matter, but Descartes is clear in his saying that quantity or number is conceptually distinct from extended substance.

[8] AT VIIIA 45; CSM I 226–7.

On this front, Leibniz's view seems friendly to Descartes's. This, even though Descartes will hold that extension exists independently of mind, and Leibniz will hold (in his later work) that extension arises from the monads (minds), and so is not independent of mind. However, relying on De Risi's reading of Leibniz's conception of extension as an emerging relational nexus holding between points (the monads), we learned that:

> Such uniformity [of the relations holding between monads] can be more or less understood as what we would today call homogeneity or, perhaps better, isotropy of space. It is the property that grants the transportation of figures through congruence, and thus free motion.[12]

Here, the relational nexus serves as medium, which not only makes way for the possibility of the *locus* of geometrical objects, but for their *movement*. As I argued in Part III, for Leibniz, a notion (or function) determines a line or geometrical shape. I called the relational nexus 'matter', which corresponds to Descartes's extension. We can now see that it does so within the context of one of the transcendental readings worked out above. In line with Mercer's view, though keeping in mind that hers was an explication of Leibniz's early work, I called that which arises from the unity of matter and form (here, form is expressed by the notion), a 'body'. Even on Leibniz's later view, matter is *prior* to a particular body (the latter understood in this context as an impenetrable geometrical shape), for the relational nexus is a necessary condition for instantiating any particular body—the nexus is that which serves to 'locate' a body.

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[9] AT VII 13; CSM II 9.

[10] Ibid.

[11] Archibald points out that number theory evolved around the concept of divisibility. Consider some early theorems: (1) Every non-zero integer *a* divides itself; that is, a/a if $a \neq 0$; and (2) Every non-zero integer *a* which divides *c* also divides every multiple of *c*; that is, if a/c, then for every k, a/kc. Ralph Archibald, *Introduction to the Theory of Numbers* (Columbus: Merrill, 1970).

[12] Vincenzo De Risi (2005), p. 130.

What is more, for Leibniz, mathematics would be impossible (that is, unintelligible) without matter (construed as that relational nexus). Even early in Leibniz's career he says that number and shape are not perfections, and so are not attributes of God. As far as I am aware, he never changes his view on this. The view suggests, I think, that in a world where God is the only thing, numbers and shapes are nowhere to be found. In other words, there is no mathematics in a world consisting solely of God. I suspect that if the monad is taken to be analogous to the Euclidean point, and God creates a multitude of monads (though it will be a category mistake to take multitude here as quantitative), *and* a relational nexus emerges necessarily among them, matter can be understood as arising from God's creating the multitude of monads (minds). The 'division' of this relational nexus, the nexus constituting extension, is what produces quantity or number. So, although the monads are a necessary part of this story, the *necessity* of their being related cannot be ignored. As he points out in the *Monadology*, the very possibility of their being related is inherent in their being distinct.[13] That is, the relational nexus is inherent in any *plurality* of monads.[14] The nexus's 'emerging' from the plurality is logically necessary. It is an all or nothing affair: in creating even one monad (analogous to a point), God creates the multitude of monads (which, according to the analogy, gives rise immediately to the plane—extension). Even so, this plurality is not coextensive with quantity. Quantity emerges in the *division* of extension or matter (the relational nexus), the division determined by a notion (which, we could say, defines a motion, which in part determines the individuated, impenetrable, shaped-body).

Now, although not entirely clear in Descartes's work, it was clear in Leibniz's that synthesis was understood in terms of combinatorics. In Part III, Leibniz, Pascal, and others were shown to have been interested in working out the mathematics underwriting the art of combinations. Here, where it was shown that even Descartes would seem to be in agreement, if it turns out that God creates two or more bodies (by dividing the plenum), where the bodies are combinatorial in nature, God looks to have created a permutation group. And, in so doing, God can be understood as having created all the conditions necessary (and sufficient) for mathematics. This, I argued, in part established the conditional:

If matter exists, mathematics is intelligible.

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[13] *Monadology*, Proposition 13.

[14] This reminds me of Galileo's view in the *Discoursi*. As I suggested in Part I, recall, Galileo says that infinity, which here I am associating with this non-quantitative plurality, is *prior* in some important sense to quantity. Galileo Galilei, *Two New Sciences*, translated by Stillman Drake (Madison: University of Wisconsin Press, 1974), pp. 38–41.

Or, perhaps better put, the examination of combinatorics pointed the way towards understanding that, in creating a material cosmos, God creates a mathematical one. When conjoining this conditional with the other, then, we get:

Mathematics is intelligible if, and only if, matter exists.

Lastly, reflecting on Parts II and III, specifically echoing the closing remarks of Part III, one of the more important discoveries in our exploration is the hidden identity (mathematically speaking) of analysis and synthesis. We found that the set {[0], [1], [2]}, for example, was an enumeration of the integers, where the equivalence relation that sorted them out into the three equivalence classes was the mathematical operator $\oplus$. The operator, in other words, turns out to be an equivalence relation. Additionally, when combined by way of this binary operator, the combinations of the members of this set form a *system*, which turns out to express the four conditions underwriting a group. As we learned, the group concept in mathematics expresses the very conditions underwriting *all* of mathematics. An enumeration, then, is isomorphic to a group (that is, it *is* a group). As worked out in Part II, the enumeration arises by way of the method of analysis. As worked out in Part III, understood as a combinatorial system, the enumeration (with its defining binary operator) proves to be a group, a genuine mathematical system. Since the group concept is what accounts for *additivity*, and additivity is what is needed in a mathematical physics, analysis and synthesis can be seen as playing an important part (here I am speaking of conceptual importance) in the development of a 'mathematized' or mathematical physics. This was brought more clearly to light when we understood how analysis and synthesis worked towards the making of equations in physics.

And so, to answer the sort of question that Aristes raised, 'Why matter?', we now have something on which to 'hang our hat'. Matter is what underwrites the intelligibility (that is, the possibility) of mathematics. And so, if God wants a mathematical reality, matter will be necessarily in the mix. It is by way of his creating matter that such a reality (that is, a mathematical reality) is created. Matter, then, seems to have been taken by seventeenth-century philosophers, such as Descartes and Leibniz, to serve as the metaphysical medium by way of which the truth (and hence the meaning) of mathematics is instantiated by God. And so, what Aristes should see, now that we have arrived at the end of our examination, is that given that God creates a mathematical reality, and that matter is the medium by way of which this reality is manifested, matter matters.

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'*Matter Matters* is a work of genius. The work exhibits a breathtaking spread of erudition from antiquity to the present, mobilized to elucidate the early modern significance of the concept of matter. The slight play of words in the title expresses the principal thesis of the work, that mathematics is intelligible for Descartes if and only if matter exists as its object. Smith understands, better than anyone, how Descartes could claim, literally, that "my physics is nothing but geometry." Many will be convinced, some dismayed, and all will be dazzled by this book.'

Thomas M. Lennon, *The University of Western Ontario*

Why is there a material world? Why is it fundamentally mathematical? *Matter Matters* explores a seventeenth-century answer to these questions as it emerged from the works of Descartes and Leibniz. The 'mathematization' of the physics is shown to have been conceptually underwritten by two methods of philosophizing, namely, analysis and synthesis. The connection between these things—mathematics, matter, and the methods of analysis and synthesis—has thus far gone unexplored by scholars. The book is in four parts: Part I works out the context in which the theory of modern matter arose. Part II develops the method of analysis, showing how it aligns with Descartes's famous doctrine of clear and distinct ideas. Part III develops the method of synthesis, focusing primarily on Leibniz, showing how it establishes the very conditions necessary and sufficient for mathematics. Analysis and synthesis turn out to establish isomorphic conceptual systems, which turn out to be isomorphic to what mathematicians today call a *group*. The group concept expresses the conditions underwriting all of mathematics. Part IV examines several relatively new interpretations of Descartes—the realist and idealist readings—which appear to be at odds with one another. The examination shows the sense in which these readings are actually compatible, and together reveal a richer picture of Descartes's position on the reality of matter. Ultimately, *Matter Matters* establishes the claim that *mathematics is intelligible if, and only if, matter exists.*

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